Castelnuovo-Mumford Regularity and Computing the de Rham Cohomology of Smooth Projective Varieties

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Abstract

We describe a parallel polynomial time algorithm for computing the topological Betti numbers of a smooth complex projective variety X. It is the first single exponential time algorithm for computing the Betti numbers of a significant class of complex varieties of arbitrary dimension. Our main theoretical result is that the Castelnuovo-Mumford regularity of the sheaf of differential p-forms on X is bounded by $p(em+1)D$, where e, m, and D are the maximal codimension, dimension, and degree, respectively, of all irreducible components of X. It follows that, for a union V of generic hyperplane sections in X, the algebraic de Rham cohomology of $X \setminus V$ is described by differential forms with poles along V of single exponential order. By covering X with sets of this type and using a Čech process, this yields a similar description of the de Rham cohomology of X, which allows its efficient computation. Furthermore, we give a parallel polynomial time algorithm for testing whether a projective variety is smooth.

Key words Castelnuovo-Mumford regularity, de Rham cohomology, algorithm, complexity, parallel polynomial time, smooth projective variety, Betti numbers, Cech cohomology, hypercohomology

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1 Introduction

A long standing open problem in algorithmic real algebraic geometry is to construct a single exponential time algorithm for computing the Betti numbers of semialgebraic sets (for an overview see [4]). The best result in this direction

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is in [3] saying that for fixed ℓ one can compute the first ℓ Betti numbers of a semialgebraic set in single exponential time.

In the complex setting one approach for computing Betti numbers is to compute the algebraic de Rham cohomology. A result of Grothendieck [30] states that the de Rham cohomology of a smooth complex variety is canonically isomorphic to the singular cohomology. An algorithm based on $\mathcal{D}$-modules for computing the de Rham cohomology of the complement of a complex affine variety is given in [46, 61]. This algorithm is used in [62] to compute the cohomology of a projective variety. However, these algorithms are not analyzed with regard to their complexity. Furthermore, they use Gröbner basis computations in a non-commutative setting, so that a good worst-case complexity is not to be expected. Indeed, already in the commutative case, computing Gröbner bases is exponential space complete [40, 39].

Via the well known Hodge decomposition, the de Rham cohomology of a smooth projective variety X is related to the sheaf cohomologies of the sheaves of differential forms on X. Algorithms for computing sheaf cohomology are described in [58, 54, 19] and implemented in Macaulay2 [28], see also [21].

A parallel polynomial time algorithm for counting the connected components (i.e., computing the zeroth Betti number) of a complex affine variety is given in [12]. Although this problem can also be solved by applying the corresponding real algorithms, the algorithm of [12] is the first one using the field structure of $\mathbb{C}$ only. It also extends to counting the irreducible components. In [52] it is described how one can compute equations for the components in parallel polynomial time.

Concerning lower bounds it is shown in [51] that it is PSPACE-hard to compute some fixed Betti number of a complex affine or projective variety given over the integers. Note that the varieties constructed in this reduction are highly singular. We do not know of any lower bound result for the problem of testing whether a variety is smooth.

1.1 Main Result

In this paper we describe an algorithm for computing the algebraic de Rham cohomology of a smooth projective variety running in parallel polynomial time. It is based on the same techniques as the algorithm in [12] using squarefree regular chains. Namely, by applying the algorithm of Szántó [56, 57] we can construct a linear system of equations describing the ideal (up to a given degree) of an affine variety. This allows to compute a linear system describing the vanishing of differential forms of some degree. Given a smooth projective variety $X \subseteq \mathbb{P}^n$ of dimension m and generic hyperplane sections $H_0, \ldots, H_m \subseteq X$ with $H_0 \cap \cdots \cap H_m = \emptyset$, their complements $U_i = X \setminus H_i$ form an open affine cover of X. Under the additional assumption that the hypersurface $H_0 \cup \cdots \cup H_m$ has normal crossings, we are able to compute the cohomologies of the affine patches $U_{i_0 \cdots i_q} = U_{i_0} \cap \cdots \cap U_{i_q}$ and by a Čech process also the cohomology of X.

More precisely, let D be the maximal degree and e the maximal codimension of all irreducible components of X, and choose $s \geq m(em+1)D$. Now consider the double complex

$$K^{p,q} := \bigoplus_{i_0<\cdots<i_q} \Gamma(X, \Omega_X^p((s+p)(H_{i_0} \cup \cdots \cup H_{i_q})))$$

together with the Čech differential $\delta\colon K^{p,q} \to K^{p,q+1}$ and the exterior differential $\mathrm{d}\colon K^{p,q} \to K^{p+1,q}$. We show that the de Rham cohomology of X can be computed as the cohomology of the total complex of $K^{\bullet,\bullet}$, i.e.,

$$H^\bullet_{\mathrm{dR}}(X) \simeq H^\bullet(\mathrm{tot}^\bullet(K^{\bullet,\bullet})).$$

To describe the output of the algorithm explicitly, let the H_i be defined by the linear forms ℓ_i. The cohomology $H^k_{\mathrm{dR}}(X)$ is then represented by a basis consisting of vectors of rational differential forms $\omega = (\omega_{i_0\cdots i_q})$ of the form

$$\omega_{i_0\cdots i_q} = \frac{1}{(\ell_{i_0}\cdots\ell_{i_q})^t} \sum_{0\leq j_1<\cdots<j_p\leq n} \omega_{i_0\cdots i_q}^{j_1\cdots j_p} \mathrm{d}X_{j_1} \wedge \cdots \wedge \mathrm{d}X_{j_p}, \qquad (1)$$

where $p+q=k$, $t \leq s+p$, and the polynomials $\omega_{i_0\cdots i_q}^{j_1\cdots j_p} \in \mathbb{C}[X_0,\ldots,X_n]$ are homogeneous of degree $t(q+1)-p$.

Furthermore, we show how to test whether a variety X is smooth and how to choose sufficiently generic generic hyperplanes H_i with normal crossings in parallel polynomial time. In summary, we prove the following theorem.

Theorem 1.1. *Given homogeneous polynomials $f_1,\ldots,f_r \in \mathbb{C}[X_0,\ldots,X_n]$ of degree at most d, one can test whether $X := \mathcal{Z}(f_1,\ldots,f_r) \subseteq \mathbb{P}^n$ is smooth and if so, compute the algebraic de Rham cohomology $H^\bullet_{\mathrm{dR}}(X)$ in parallel time $(n\log d)^{\mathcal{O}(1)}$ and sequential time $d^{\mathcal{O}(n^4)}$.*

As for the necessity of squarefree regular chains in our algorithm we remark that for smooth varieties there are single exponential bounds for the degrees in a Gröbner basis (cf. §1.2). So perhaps one could replace the squarefree regular chains in our approach by Gröbner bases. But we do not know whether such an algorithm would be well-parallelizable.

The new aspect of our result compared to the methods of [46, 61, 62] is the complexity analysis of the algorithm, which is the main purpose of this paper. We do not expect that our algorithm as stated would yield a practically efficient implementation. However, it seems conceivable that a variation of the approach (perhaps using Gröbner bases) could produce a reasonable implementiation.

1.2 Castelnuovo-Mumford Regularity

The main theoretical result of this paper is a bound on the Castelnuovo-Mumford regularity of the sheaf of regular differential p-forms on a smooth projective variety X. This result allows to bound the degrees of the differential forms one has to deal with in computing the cohomology of X. More precisely, the regularity yields a bound on the order t in (1), which in turn determines the degree of the coefficients.

The Castelnuovo-Mumford regularity was defined in [43] for sheaves on $\mathbb{P}^n$. The definition has been modified in [20] to apply to a homogeneous ideal I. This notion was related to computational complexity in [6] by showing that the regularity of I equals the maximal degree in a reduced Gröbner basis of I with respect to the degree reverse lexicographic order in generic coordinates. In this respect upper bounds on the regularity of a homogeneous ideal are of particular interest in computational algebraic geometry and commutative algebra. For a general ideal, double exponential upper bounds were shown in [26] and [24]. The famous example of [40] shows that this is essentially best possible. However, there are several results giving better bounds for the regularity in special cases, such as [31, 48, 37, 49, 47, 55, 35, 25]. A nice overview over these kinds of results is given in [5]. This paper also contains a bound on the regularity of the ideal of a smooth veriety X, which is asymptotic to the product of the degree and the dimension of X. A more precise bound in terms of the degrees of generators of the ideal of X is proved in [8]. More generally, the authors prove the vanishing of the higher cohomology of powers of the sufficiently twisted ideal sheaf of X (cf. Proposition 3.6). Our bound on the regularity of the sheaf of differential forms on X is deduced from this result. We are not aware of any other bounds on the regularity of a sheaf other than a power of an ideal sheaf.

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2 Preliminaries

2.1 Basic Notations

Denote by $\mathbb{P}^n := \mathbb{P}^n(\mathbb{C})$ the projective space over $\mathbb{C}$. A *(closed) projective variety* $X \subseteq \mathbb{P}^n$ is defined as the zero set

$$X = \mathcal{Z}(f_1 \dots, f_r) := \{x \in \mathbb{P}^n \mid f_1(x) = \cdots = f_r(x) = 0\}$$

of homogeneous polynomials $f_1 \dots, f_r \in \mathbb{C}[X_0, \dots, X_n]$. Note that X may be reducible. Occasionally we write $\mathcal{Z}_X(f) := X \cap \mathcal{Z}(f)$ for $f \in \mathbb{C}[X_0, \dots, X_n]$. A *quasi-projective variety* is a difference $X \setminus Y$, where X and Y are closed projective varieties. The term variety will always mean quasi-projective variety. The *homogeneous (vanishing) ideal* $I(X)$ of the variety X is defined as the ideal generated by the homogeneous polynomials vanishing on X. The *homogeneous coordinate ring* of X is $\mathbb{C}[X] := \mathbb{C}[X_0, \ldots, X_n]/I(X)$. By the (weak) homogeneous Nullstellensatz we have $\mathcal{Z}(f_1, \ldots, f_r) = \emptyset$ iff there exists $N \in \mathbb{N}$ with $(X_0, \ldots, X_n)^N \subseteq (f_1, \ldots, f_r)$. Its effective version states that one can choose $N = (n+1)d - n$, when $\deg f_i \leq d$ [36, Théorème 3.3]. According to the affine effective Nullstellensatz, for $f_1, \ldots, f_r \in \mathbb{C}[X_1, \ldots, X_n]$ of degree $\leq d$, we have $\mathcal{Z}(f_1, \ldots, f_r) = \emptyset$ iff there exist polynomials $g_1, \ldots, g_r$ with $\deg(g_i f_i) \leq d^n$ such that $1 = \sum_i g_i f_i$ [10, 34, 22, 33].

The *dimension* $\dim X$ is the *Krull dimension* of X in the Zariski topology. A variety all of whose irreducible components have the same dimension m is called *(m-)equidimensional*. The *local dimension* $\dim_x X$ at $x \in X$ is defined as the maximal dimension of all components through x. A *hypersurface* of a variety X is a closed subvariety V with $\dim_x V = \dim_x X - 1$ for all $x \in V$.

We often identify $\mathbb{P}^n \setminus \mathcal{Z}(X_i) \simeq \mathbb{C}^n$, $0 \leq i \leq n$, via $(x_0 : \cdots : x_n) \mapsto (\frac{x_0}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i})$, where as usual $\widehat{\ }$ denotes omission. Under this identification, a homogeneous polynomial $f \in \mathbb{C}[X_0, \ldots, X_n]$ corresponds to its dehomogenization f^i by setting $X_i := 1$. We thus get a surjection ${}^i \colon \mathbb{C}[X_0, \ldots, X_n] \to \mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]$, and the image of $I(X)$ under this map is the vanishing ideal of the affine variety $X \setminus \mathcal{Z}(X_i)$. Now let $f_1, \ldots, f_r$ be homogeneous polynomials defining the hypersurfaces $H_1, \ldots, H_r$ of $\mathbb{P}^n$. Then we say that the closed variety X is *scheme-theoretically cut out by the hypersurfaces* $H_1, \ldots, H_r$ iff for each i, the dehomogenizations $f_1^i, \ldots, f_r^i$ generate the image of $I(X)$ in $\mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]$.

For a polynomial $f \in \mathbb{C}[X_1, \ldots, X_n]$ its *differential* at $x \in \mathbb{C}^n$ is the linear function $d_x f \colon \mathbb{C}^n \to \mathbb{C}$ defined by $d_x f(v) := \sum_i \frac{\partial f}{\partial X_i}(x) v_i$. The tangent space of the variety X at $x \in X \setminus \mathcal{Z}(X_i)$ is defined as the vector subspace

$$T_x X := \{v \in \mathbb{C}^n \mid \forall f \in I(X)\ d_x f^i(v) = 0\} \subseteq \mathbb{C}^n.$$

If X is scheme-theoretically cut out by the hypersurfaces defined by the homogeneous polynomials $f_1, \ldots, f_r$, then $T_x X = \mathcal{Z}(d_x f_1^i, \ldots, d_x f_r^i)$. We have $\dim T_x X \geq \dim_x X$ for all $x \in X$. We say that $x \in X$ is a *smooth* point in X iff $\dim T_x X = \dim_x X$. The variety X is *smooth* iff all of its points are smooth.

The *degree* $\deg X$ of an irreducible closed variety X of dimension m is defined as the maximal cardinality of $X \cap L$ over all linear subspaces $L \subseteq \mathbb{P}^n$ of dimension $n - m$ [45, §5A]. We define the (cumulative) degree $\deg X$ of a reducible variety X to be the sum of the degrees of *all* irreducible components of X. It follows essentially from Bézout's Theorem that if X is defined by polynomials of degree $\leq d$, then $\deg X \leq d^n$ [14].

2.2 Coherent Sheaves

Let X be a closed variety in $\mathbb{P}^n$. Then every graded $\mathbb{C}[X]$-module M gives rise to a sheaf $\widetilde{M}$ of $\mathcal{O}_X$-modules on X such that, on a principal open set $U = X \setminus \mathcal{Z}(f)$, the sections of $\widetilde{M}$ are given by $\Gamma(U, \widetilde{M}) = M_{(f)}$, the degree 0 part of the localization of M at f. A sheaf $\mathcal{F}$ on X is called *coherent* iff $\mathcal{F} = \widetilde{M}$ with a finitely generated graded $\mathbb{C}[X]$-module M.

An important example is of course the structure sheaf $\mathcal{O}_X = \widetilde{\mathbb{C}[X]}$. We also define the *twisting sheaf* $\mathcal{O}_X(k) := \widetilde{\mathbb{C}[X](k)}$ for $k \in \mathbb{Z}$, where $\mathbb{C}[X](k)_d := \mathbb{C}[X]_{k+d}$. The twisting sheaf behaves well with respect to direct and inverse images under maps. In particular, for a closed embedding $i\colon X \hookrightarrow Y$ of projective varieties we have $i^*(\mathcal{O}_Y(k)) \simeq \mathcal{O}_X(k)$ and $i_*(\mathcal{O}_X(k)) \simeq (i_*\mathcal{O}_X)(k)$. Furthermore, we have $\mathcal{O}_X(k)\otimes\mathcal{O}_X(\ell) \simeq \mathcal{O}_X(k+\ell)$ [32, II, Proposition 5.12]. The sheaf $\mathcal{O}_X(1)$ is called the *very ample line bundle* on X determined by the embedding $X \hookrightarrow \mathbb{P}^n$. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X we define the *twisted* sheaf $\mathcal{F}(k) := \mathcal{F} \otimes \mathcal{O}_X(k)$ for $k \in \mathbb{Z}$. The ideal sheaf $\mathcal{I}_X$ of X is defined as the kernel of the restriction map $\mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_X$. We have $\mathcal{I}_X = \widetilde{I(X)}$, hence the ideal sheaf is coherent.

Fundamental for us is the sheaf of *regular differential forms*, which is defined as follows. Let $\mathcal{U} := \{U_i \mid 0 \le i \le s\}$ be an open affine cover of X (we can take $U_i := \{X_i \neq 0\}$, but any other open cover works too). Denote by $\mathbb{C}[U_i]$ the affine coordinate ring of U_i, and let Ω_{U_i} be the $\mathbb{C}[U_i]$-module of *Kähler differentials* $\Omega_{\mathbb{C}[U_i]/\mathbb{C}}$ [18, Chapter 16]. Then, Ω_X is defined as the sheaf on X obtained by glueing the sheaves on U_i corresponding to Ω_{U_i}. This means that one determines a section s on an open set $U \subseteq X$ by giving for each i a section $s_i \in \Omega_{U_i \cap U}$ with the property that for all i, j we have $s_i|U_i{\cap}U_j{\cap}U = s_j|U_i{\cap}U_j{\cap}U$. The universal derivations $\mathrm{d}\colon \mathbb{C}[U_i] \to \Omega_{U_i}$ glue together to give a map $\mathrm{d}\colon \mathcal{O}_X \to \Omega_X$ of sheaves, which is a derivation on the stalks. The p-fold exterior power $\Omega_X^p := \bigwedge^p \Omega_X$ is called the sheaf of *regular (p-)forms*. The derivation $\mathrm{d}\colon \mathcal{O}_X \to \Omega_X$ uniquely extends to the *exterior derivative* $\mathrm{d}\colon \Omega_X^p \to \Omega_X^{p+1}$ satisfying Leibnitz' rule and $d \circ d = 0$, so that we get the *(algebraic) de Rham complex*

$$\Omega_X^\bullet\colon\ 0 \longrightarrow \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Omega_X^m \longrightarrow 0,$$

where $m = \dim X$.

For $X = \mathbb{P}^n$ there is a more explicit description of this complex, which we give now (cf. §6.1 of [17]). Denote by Λ the module of Kähler differentials $\Omega_{\mathbb{C}[X_0,\dots,X_n]/\mathbb{C}}$, which is the free module generated by $\mathrm{d}X_0, \dots, \mathrm{d}X_n$ [18, Proposition 16.1]. The universal derivation $\mathrm{d}\colon \mathbb{C}[X_0, \dots, X_n] \to \Lambda$ is given by $\mathrm{d}f = \sum_i \frac{\partial f}{\partial X_i}\mathrm{d}X_i$. Then, $\Lambda^p := \bigwedge^p \Lambda$ is the free module generated by $\mathrm{d}X_{i_1} \wedge \cdots \wedge \mathrm{d}X_{i_p}$, $0 \le i_1 < \cdots < i_p \le n$. Furthermore, the exterior derivative $\mathrm{d}\colon \Lambda^p \to \Lambda^{p+1}$ yields the *de Rham complex*

$$\Lambda^0 = \mathbb{C}[X_0, \dots, X_n] \xrightarrow{\mathrm{d}} \Lambda^1 \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} \Lambda^n \xrightarrow{\mathrm{d}} \Lambda^{n+1}$$

The modules Λ^p are graded by setting

$$\deg(g\mathrm{d}X_{i_1} \wedge \cdots \wedge \mathrm{d}X_{i_p}) := \deg(g) + p, \quad g \in \mathbb{C}[X_0, \dots, X_n] \quad \text{homogeneous}.$$

Then d is a map of degree 0. There is another derivation $\Delta\colon \Lambda^p \to \Lambda^{p-1}$ of degree 0, which can be defined as the *contraction with the Euler vector field*

$\sum_i X_i \frac{\partial}{\partial X_i}$. It is uniquely determined by Leibnitz' rule and the formula $\Delta(\mathrm{d}f) = \deg f \cdot f$ for a homogeneous polynomial f, and satisfies $\Delta(\mathrm{d}\alpha)+\mathrm{d}(\Delta\alpha) = \deg\alpha\cdot\alpha$ for any homogeneous $\alpha \in \Lambda^p$.

Now put $M^p := \ker(\Delta\colon \Lambda^p \to \Lambda^{p-1})$. One can define the sheaf of differential p-forms on $\mathbb{P}^n$ by setting $\Omega^p_{\mathbb{P}^n} := \widetilde{M^p}$. Hence, for a homogeneous polynomial f of degree k, each differential p-form on $\mathbb{P}^n \setminus \mathcal{Z}(f)$ is of the form

$$\omega = \frac{\alpha}{f^t} \quad \text{with} \quad \deg\alpha = tk \quad \text{and} \quad \Delta(\alpha) = 0, \tag{2}$$

where $\alpha \in \Lambda^p$ is homogeneous and $t \in \mathbb{N}$. By the usual quotient rule one can extend $\mathrm{d}\colon \Lambda^p \to \Lambda^{p+1}$ to localizations. Then one easily checks that $\mathrm{d}(M^p_{(f)}) \subseteq M^{p+1}_{(f)}$ for homogeneous $f \in \mathbb{C}[X_0, \dots, X_n]$. This defines the exterior derivative $\mathrm{d}\colon \Omega^p_{\mathbb{P}^n} \to \Omega^{p+1}_{\mathbb{P}^n}$ on the sheaf level.

A sheaf $\mathcal{F}$ on X is said to be *locally free* iff it is locally isomorphic to a direct sum of copies of $\mathcal{O}_X$. The local rank of $\mathcal{F}$ is the number of copies of the structure sheaf needed, which is a locally constant function. If X is smooth and m-equidimensional, then Ω^p_X is locally free of rank $\binom{m}{p}$.

The isomorphism classes of locally free sheaves on X of rank k are in one-to-one correspondence with those of vector bundles over X of rank k [32, II, Exercise 5.18]. A locally free sheaf of rank 1 is called an *invertible sheaf* or *line bundle*. The tensor product $\mathcal{L} \otimes \mathcal{M}$ of two line bundles $\mathcal{L}, \mathcal{M}$ is also a line bundle. For any line bundle $\mathcal{L}$, its *dual* sheaf $\check{\mathcal{L}} := \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$ is another line bundle satisfying $\mathcal{L} \otimes \check{\mathcal{L}} \simeq \mathcal{O}_X$ [32, II, Proposition 6.12]. In particular, the sheaves $\mathcal{O}_X(k)$ defined above are line bundles, and we have $\mathcal{O}_X(k)\check{} = \mathcal{O}_X(-k)$.

2.3 Divisors and Line Bundles

Let X be a smooth variety. A divisor on X is an element of the free abelian group $\operatorname{Div} X$ generated by the irreducible hypersurfaces of X. This means that each $D \in \operatorname{Div} X$ is a formal linear combination $D = \sum_i m_i V_i$, where $m_i \in \mathbb{Z}$ and $V_i \subseteq X$ are irreducible hypersurfaces. The *support* of D is defined as $\operatorname{supp} D := \bigcup_{m_i \neq 0} V_i$. Each $D \in \operatorname{Div} X$ defines a line bundle $\mathcal{O}_X(D)$, which is a subsheaf of the sheaf $\mathcal{K}$ *of total quotient rings of* $\mathcal{O}_X$ [43, p. 61]. The stalk $\mathcal{O}_X(D)_x$ is $\mathcal{O}_{X,x}$, if $x \notin \operatorname{supp} D$. For $x \in \operatorname{supp} D$, the stalk is $\prod_i f_i^{-m_i}\mathcal{O}_{X,x}$, where f_i is a local equation of V_i at x. The rule $D \mapsto \mathcal{O}_X(D)$ maps $\operatorname{Div} X$ bijectively onto the invertible subsheaves of $\mathcal{K}$, and it maps sums to tensor products [32, II, Proposition 6.13].

Let X be closed and H a hyperplane in $\mathbb{P}^n$ meeting X *properly*, i.e., H does not contain any irreducible component of X, so that $X \cap H$ is a hypersurface in X. Then the *hyperplane section* $V := X \cap H$ defines a divisor $H \cdot X = \sum_i m_i V_i$, where the V_i are the irreducible components of V, and m_i is the *intersection multiplicity* $i(X, H; V_i)$ between X and H along V_i, which can be defined as follows. Choose $x \in V_i$ and a (reduced) local equation $f \in \mathcal{O}_{\mathbb{P}^n,x}$ of H. Then $i(X, H; V_i)$ is the *order of vanishing* $\operatorname{ord}_{V_i}(f)$ of f along V_i, i.e., the maximal $k \in \mathbb{N}$ such that $f = gh^k$ with some g in $\mathcal{O}_{X,x}$, where h is a local equation of V_i

at x. Note that $\mathcal{O}_{X,x}$ is factorial, since X is smooth. The line bundle $\mathcal{O}_X(H\cdot X)$ is isomorphic to the very ample line bundle $\mathcal{O}_X(1)$.

A hypersurface $V \subseteq X$ is said to have *normal crossings* iff for each $x \in V$ contained in k irreducible components $V_1, \ldots, V_k$ of V, there exist local equations $f_i \in \mathcal{O}_{X,x}$ of V_i around x, such that $d_x f_1, \ldots, d_x f_k$ are linearly independent in the dual $(T_x X)^*$. Note that the case $k = 1$ implies that all irreducible components of V are smooth.

2.4 Sheaf Cohomology

Let $\mathcal{F}$ be a coherent sheaf and $\mathcal{U} := \{U_i \,|\, 0 \le i \le s\}$ an open cover of the variety X. For $0 \le q \le s$ and $0 \le i_0 < \cdots < i_q \le s$ set $U_{i_0\cdots i_q} := U_{i_0} \cap \cdots \cap U_{i_q}$. The *Čech complex* is defined by $C^q := C^q(\mathcal{U}, \mathcal{F}) := \bigoplus_{i_0<\cdots<i_q} \mathcal{F}(U_{i_0\cdots i_q})$, with the *Čech differential* $\delta\colon C^q \to C^{q+1}$ given by

$$(\delta(\omega))_{i_0\cdots i_{q+1}} := \sum_{\nu=0}^{q+1} (-1)^\nu \omega_{i_0\cdots\widehat{i_\nu}\cdots i_{q+1}} | U_{i_0\cdots i_{q+1}} \quad \text{for} \quad \omega = (\omega_{i_0\cdots i_q}) \in C^q. \quad (3)$$

Then one easily checks that $\delta \circ \delta = 0$, hence $(C^\bullet, \delta)$ is indeed a complex. Its cohomology $H^i(\mathcal{U}, \mathcal{F}) := H^i(C^\bullet, \delta)$ is called the i-th *Čech cohomology* of $\mathcal{F}$ with respect to $\mathcal{U}$. The i-th *Čech cohomology* (or *sheaf cohomology*) of $\mathcal{F}$ is defined as the direct limit over all open covers $\mathcal{U}$ of X, directed by refinements. A sheaf $\mathcal{F}$ on X is called *acyclic* iff $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

A cover $\mathcal{U}$ of X is called a *Leray cover* for $\mathcal{F}$ iff $\mathcal{F}$ is acyclic on $U_{i_0\cdots i_q}$ for all $i_0 < \cdots < i_q$. Leray's Theorem states that in this case we have $H^i(X, \mathcal{F}) = H^i(\mathcal{U}, \mathcal{F})$ [32, III, Exercise 4.11]. Since $\mathcal{F}$ is a coherent sheaf, this is true for any *affine* cover [32, III, Theorem 3.5]. It easily follows that for a morphism $f\colon X \to Y$ there is a natural isomorphism $H^i(X, \mathcal{F}) \simeq H^i(Y, f_*\mathcal{F})$, where $f_*\mathcal{F}$ denotes the direct image of the sheaf $\mathcal{F}$ under f [32, III, Exercise 4.1].

2.5 Hypercohomology and de Rham Cohomology

The material in this section is explained e.g. in [29]. Let X be a smooth variety and consider a complex of coherent sheaves $(\mathcal{F}^\bullet, \mathrm{d})$ on X with $\mathcal{F}^p = 0$ for $p < 0$. Then, for an open cover $\mathcal{U}$, the Čech complexes $C^\bullet(\mathcal{U}, \mathcal{F}^p)$ as defined in §2.4 fit together to the *Čech double complex* $C^{\bullet,\bullet} := C^{\bullet,\bullet}(\mathcal{U}, \mathcal{F}^\bullet)$ by setting

$$C^{p,q}(\mathcal{U}, \mathcal{F}^\bullet) = \bigoplus_{i_0<\cdots<i_q} \mathcal{F}^p(U_{i_0\cdots i_q}) \quad \text{for all} \quad p, q \ge 0.$$

The two differentials are the one induced by the differential d of $\mathcal{F}$ and the Čech differential δ defined by (3). Denote by $\mathbb{H}^\bullet(\mathcal{U}, \mathcal{F}^\bullet) := H^\bullet(\mathrm{tot}^\bullet(C^{\bullet,\bullet}))$ the cohomology of the total complex of $C^{\bullet,\bullet}(\mathcal{U}, \mathcal{F}^\bullet)$. Then the *hypercohomology* $\mathbb{H}^i(X, \mathcal{F}^\bullet)$ of the complex of sheaves $\mathcal{F}^\bullet$ is defined as the direct limit of $\mathbb{H}^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ over all open covers $\mathcal{U}$ of X, directed by refinement. As for any double complex [41, §2.4], there are two spectral sequences

$$
\begin{aligned}
{}'E_2^{p,q} = H_{\mathrm{d}}^p(H^q(X,\mathcal{F}^\bullet)) \quad &\Rightarrow \quad \mathbb{H}^{p+q}(X,\mathcal{F}^\bullet) \quad \text{and} \\
{}''E_2^{p,q} = H^q(X,\mathcal{H}^p(\mathcal{F}^\bullet)) \quad &\Rightarrow \quad \mathbb{H}^{p+q}(X,\mathcal{F}^\bullet),
\end{aligned}
$$

where the *cohomology sheaf* is defined by

$$
\mathcal{H}^p(\mathcal{F}^\bullet) := \ker(\mathrm{d}\colon \mathcal{F}^p \to \mathcal{F}^{p+1})/\mathrm{im}\,(\mathrm{d}\colon \mathcal{F}^{p-1} \to \mathcal{F}^p).
$$

The first spectral sequence implies that if all the sheaves $\mathcal{F}^p$ are acyclic, then $\mathbb{H}^\bullet(X,\mathcal{F}^\bullet) = H^\bullet(\Gamma(X,\mathcal{F}^\bullet))$ is the cohomology of the complex of global sections. Similarly as for sheaf cohomology we have $\mathbb{H}^i(X,\mathcal{F}^\bullet) \simeq \mathbb{H}^\bullet(\mathcal{U},\mathcal{F}^\bullet)$, if $\mathcal{U}$ is a Leray cover for all $\mathcal{F}^p$.

A map of complexes of sheaves $f\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is called a *quasi-isomorphism* iff it induces an isomorphism $\mathcal{H}^\bullet(\mathcal{F}^\bullet) \to \mathcal{H}^\bullet(\mathcal{G}^\bullet)$. By comparing the second spectral sequences of the hypercohomologies of $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ it follows that f induces an isomorphism $\mathbb{H}^\bullet(X,\mathcal{F}^\bullet) \xrightarrow{\sim} \mathbb{H}^\bullet(X,\mathcal{G}^\bullet)$.

The *algebraic de Rham cohomology* of X is defined as the hypercohomology

$$
H^\bullet_{\mathrm{dR}}(X) := \mathbb{H}^\bullet(X,\Omega^\bullet_X)
$$

of the de Rham complex. If X is affine, then $H^\bullet_{\mathrm{dR}}(X)$ can be computed by taking the cohomology of the complex $\Gamma(X,\Omega^\bullet_X)$ of global sections, since all Ω^p_X are acyclic.

2.6 Computational Model

Our model of computation is that of algebraic circuits over $\mathbb{C}$, cf. [60, 11]. We set $\mathbb{C}^\infty := \bigsqcup_{n\in\mathbb{N}} \mathbb{C}^n$. The *size* of an algebraic circuit $\mathcal{C}$ is the number of nodes of $\mathcal{C}$, and its *depth* is the maximal length of a path from an input to an output node. We say that a function $f\colon \mathbb{C}^\infty \to \mathbb{C}^\infty$ can be computed *in parallel time $d(n)$ and sequential time $s(n)$* iff there exists a polynomial-time uniform family of algebraic circuits $(\mathcal{C}_n)_{n\in\mathbb{N}}$ over $\mathbb{C}$ of size $s(n)$ and depth $d(n)$ such that $\mathcal{C}_n$ computes $f|\mathbb{C}^n$.

2.7 Efficient Parallel Linear Algebra

We use differential forms to reduce our problem to linear algebra, for which efficient parallel algorithms exist. In particular, we need to be able to solve the following problems:

1. Given $A \in \mathbb{C}^{n\times m}$ and $b \in \mathbb{C}^n$, decide whether the linear system of equations $Ax = b$ has a solution and if so, compute one.

2. Compute a basis of the kernel of a matrix $A \in \mathbb{C}^{n\times m}$.

3. Compute a basis of the image of a matrix $A \in \mathbb{C}^{n\times m}$.

4. Given a linear subspace $V \subseteq \mathbb{C}^n$ in terms of a basis, and given linearly independent $v_1, \dots, v_i \in V$, extend them to a basis of V.

These problems are easily reduced to inverting a regular square-matrix (thus to computing the characteristic polynomial) and computing the rank of a matrix. For instance, the last problem boils down to rank computations as follows. Let $b_1, \dots, b_m \in V$ be the given basis. Set $B := (v_1, \dots, v_i)$. For all $j = 1, 2, \dots, m$ do: if $\mathrm{rk}\,(B, b_j) > \mathrm{rk}\, B$ then append b_j to B.

Mulmuley [42] has reduced the problem of computing the rank to the computation of the characteristic polynomial of a matrix. Since we need his construction, we describe it here. Let $A \in \mathbb{C}^{m \times m'}$ be a matrix. Then

$$\mathrm{rk} \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} = 2\,\mathrm{rk}\, A,$$

so we can assume $m = m'$. Introducing the additional variable T, define the diagonal matrix $X := \mathrm{diag}(1, T, \dots, T^{m-1})$, and consider the characteristic polynomial $p_A(Z)$ of XA over the field $\mathbb{C}(T)$, $p_A(Z) := \det(XA - ZI)$. Then the rank of A equals $m - s$, where s is the maximal integer with $Z^s | p_A(Z)$. We will call $p_A(Z)$ the *Mulmuley polynomial* of A.

The characteristic polynomial of an $m \times m$ matrix can be computed in parallel (sequential) time $\mathcal{O}(\log^2 m)$ ($m^{\mathcal{O}(1)}$) with the algorithm of [7]. If the matrix has polynomial entries of degree d in n variables, then the Berkowitz algorithm can be implemented in parallel (sequential) time $\mathcal{O}(n \log m \log(md))$ ($(md)^{\mathcal{O}(n)}$) [50].

3 Castelnuovo-Mumford Regularity

A nice exposition about various versions of Castelnuovo-Mumford regularity and vanishing results is contained in the book [38]. Let $X \subseteq \mathbb{P}^n$ be a smooth closed subvariety. Recall from §2.2 that $\mathcal{O}_X(1)$ denotes the very ample line bundle on X determined by the embedding $X \hookrightarrow \mathbb{P}^n$, and that for a coherent sheaf $\mathcal{F}$ on X we put $\mathcal{F}(k) := \mathcal{F} \otimes \mathcal{O}_X(k)$. The following definition is due to [43] building on ideas of Castelnuovo.

Definition 3.1. The coherent sheaf $\mathcal{F}$ on X is called *k-regular* iff

$$H^i\,(X, \mathcal{F}(k - i)) = 0 \quad \text{for all} \quad i > 0. \tag{4}$$

The *Castelnuovo-Mumford regularity* $\mathrm{reg}\,(\mathcal{F})$ of $\mathcal{F}$ is defined as the infimum over all $k \in \mathbb{Z}$ such that $\mathcal{F}$ is k-regular.

Remark 3.2. (i) A fundamental result of [43] is that if $\mathcal{F}$ is k-regular, then $\mathcal{F}$ is ℓ-regular for all $\ell \geq k$.

(ii) Let

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

be a short exact sequence of coherent sheaves on X. The long exact cohomology sequence shows that

$$\operatorname{reg}(\mathcal{H}) \leq \max\{\operatorname{reg}(\mathcal{F}) - 1, \operatorname{reg}(\mathcal{G})\} \quad \text{and}$$
$$\operatorname{reg}(\mathcal{G}) \leq \max\{\operatorname{reg}(\mathcal{F}), \operatorname{reg}(\mathcal{H})\}.$$

(iii) Let $i\colon X \hookrightarrow \mathbb{P}^n$ be the closed embedding, and $\mathcal{F}$ a coherent sheaf on X. The projection formula shows $i_*(\mathcal{F} \otimes i^*\mathcal{O}_{\mathbb{P}^n}(t)) = i_*\mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(t)$ for all $t \in \mathbb{Z}$ [32, II, Exercise 5.1], hence

$$\operatorname{reg}(\mathcal{F}) = \operatorname{reg}(i_*\mathcal{F}).$$

This is usually used as an argument that one can restrict to the case $X = \mathbb{P}^n$. However, we are dealing with exterior powers and thus need the more general situation, since a direct image of a locally free sheaf is not locally free, and reasonable formulas for exterior powers hold for locally free sheaves only (in particular, Corollary 3.13).

(iv) Let $X \subseteq \mathbb{P}^n$ be a subscheme of dimension m with ideal sheaf $\mathcal{I} = \mathcal{I}_X$, and let $k > 0$. Then $\mathcal{I}$ is k-regular if and only if $H^i(\mathbb{P}^n, \mathcal{I}(k-i)) = 0$ for all $0 < i \leq m+1$ [38, Example 1.8.29]. This follows from the short exact sequence $0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_X \to 0$ using that $H^i(X, \mathcal{F}) = 0$ for all $i > m$ and any coherent sheaf $\mathcal{F}$ [32, III, Theorem 2.7].

Example 3.3. (i) Theorem 5.1 in Chapter III of [32] shows $\operatorname{reg}(\mathcal{O}_{\mathbb{P}^n}) = 0$.

(ii) The structure sheaf $\mathcal{O}_X$ of a hypersurface $X \subseteq \mathbb{P}^n$ of degree D has regularity $D-1$. This follows from the isomorphism $\mathcal{I} \simeq \mathcal{O}_{\mathbb{P}^n}(-D)$ for the ideal sheaf $\mathcal{I}$ of X and the exact sequence $0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_X \to 0$.

(iii) Example (i) together with the exact sequence of [32, II, Theorem 8.13] implies $\operatorname{reg}(\Omega_{\mathbb{P}^n}) = 2$.

The aim of this section is to prove the following theorem.

Theorem 3.4. *Let $X \subset \mathbb{P}^n$ be a smooth closed subvariety of dimension m. Let D be the maximal degree and e the maximal codimension of all irreducible components of X. Then*

$$\operatorname{reg}(\Omega_X^p) \leq p(em+1)D \quad \text{for} \quad p > 0,$$
$$\operatorname{reg}(\mathcal{O}_X) \leq e(D-1).$$

Remark 3.5. For $X = \mathbb{P}^n$ the first claim is false as Example 3.3 (iii) shows.

We will reduce this theorem to the following vanishing result of [8].

Proposition 3.6. *Let $\mathcal{I}$ be the ideal sheaf of a smooth irreducible closed variety in $\mathbb{P}^n$ of codimension e, which is scheme-theoretically cut out by hypersurfaces of degrees at most D. Then*

$$H^i(\mathbb{P}^n, \mathcal{I}^a(k)) = 0 \quad \text{for} \quad a \geq 0,\ i > 0,\ k \geq (a+e-1)D - n,$$

where $\mathcal{I}^a$ denotes the a-th power of the ideal sheaf $\mathcal{I}$.

Remark 3.7. In [8] there is proved a more precise bound in terms of the individual degrees of the hypersurfaces which cut out X, but we do not need this here.

We also use the following result of [44].

Proposition 3.8. *Each smooth irreducible closed projective variety of degree D is scheme-theoretically cut out by hypersurfaces of degree D.*

Corollary 3.9. *Let $\mathcal{I}$ be the ideal sheaf of a smooth irreducible projective variety X in $\mathbb{P}^n$ of degree D and codimension e. Then*

$$\mathrm{reg}\,(\mathcal{I}^a) \le (a+e-1)D - e + 1.$$

Proof. This follows immediately from Propositions 3.6 and 3.8 together with part (iv) of Remark 3.2 (note that $\mathcal{I}^a$ is the ideal sheaf of some subscheme of the same dimension as X). □

Before we prove Theorem 3.4, let us first gather some basic properties of regularity. In the following one can always assume X to be irreducible. The next two lemmas are a version of a well known technique to characterize regularity by free resolutions.

Lemma 3.10. *Let*

$$\mathcal{F}_N \to \mathcal{F}_{N-1} \to \cdots \to \mathcal{F}_0 \to \mathcal{F} \to 0$$

be an exact sequence of coherent sheaves on X, where $N+1 \ge \dim X =: m$. Then

$$\mathrm{reg}\,(\mathcal{F}) \le \max\{\mathrm{reg}\,(\mathcal{F}_0), \mathrm{reg}\,(\mathcal{F}_1) - 1, \ldots, \mathrm{reg}\,(\mathcal{F}_{m-1}) - m + 1\}.$$

Proof. This follows easily by chasing through the complex [38, Proposition B.1.2], taking into account that $H^i(X, \mathcal{F}) = 0$ for all $i > m$ and any coherent sheaf $\mathcal{F}$. Another proof is given in [1, Lemma 3.9]. □

For a finite dimensional vector space V and a coherent sheaf $\mathcal{F}$ on X we denote by $V \otimes \mathcal{F}$ the sheaf $U \mapsto V \otimes_{\mathbb{C}} \mathcal{F}(U)$. If $v_1, \ldots, v_N$ is a basis of V, then $V \otimes \mathcal{F} = \bigoplus_{i=1}^N v_i \otimes \mathcal{F}$. The following lemma is proved as Corollary 3.2 in [1]. Set $\mathrm{Reg}\,(X) := \max\{1, \mathrm{reg}\,(\mathcal{O}_X)\}$.

Lemma 3.11. *Let $\mathcal{F}$ be a k-regular coherent sheaf on X. Then there exist finite dimensional vector spaces V_i and an exact sequence*

$$\cdots \to V_i \otimes \mathcal{O}_X(-k-iR) \to \cdots \to V_1 \otimes \mathcal{O}_X(-k-R) \to V_0 \otimes \mathcal{O}_X(-k) \to \mathcal{F} \to 0, \tag{5}$$

where $R := \mathrm{Reg}\,(X)$.

Using this we prove a bound on the regularity of tensor products.

Proposition 3.12. *Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on X, where $\mathcal{G}$ is locally free, and denote $m := \dim X$ and $R := \mathrm{Reg}\,(X)$ as above. Then*

$$\mathrm{reg}\,(\mathcal{F} \otimes \mathcal{G}) \leq \mathrm{reg}\,(\mathcal{F}) + \mathrm{reg}\,(\mathcal{G}) + (m-1)(R-1).$$

Proof. The proof parallels the one of the special case $X = \mathbb{P}^n$ [38, Proposition 1.8.9]. Let $k := \mathrm{reg}\,(\mathcal{F})$, and consider the resolution (5) of $\mathcal{F}$, which exists according to Lemma 3.11. Tensoring with $\mathcal{G}$ yields

$$\cdots \to V_i \otimes \mathcal{G}(-k-iR) \to \cdots \to V_1 \otimes \mathcal{G}(-k-R) \to V_0 \otimes \mathcal{G}(-k) \to \mathcal{F} \otimes \mathcal{G} \to 0.$$

Since tensoring with a locally free sheaf is an exact functor, this sequence is exact. Furthermore, $\mathrm{reg}\,(V_i \otimes \mathcal{G}(-k-iR)) \leq k+iR+\mathrm{reg}\,(\mathcal{G})$, hence $\mathrm{reg}\,(\mathcal{F}\otimes\mathcal{G}) \leq k + \mathrm{reg}\,(\mathcal{G}) + (m-1)(R-1)$ by Lemma 3.10. □

Corollary 3.13. *Let $\mathcal{F}$ be a locally free sheaf on X. Then for $p > 0$*

$$\mathrm{reg}\,(\Lambda^p \mathcal{F}) \leq p \cdot \mathrm{reg}\,(\mathcal{F}) + (p-1)(m-1)(R-1).$$

Proof. The same bound for the p-th tensor power of $\mathcal{F}$ clearly follows from Proposition 3.12. Since the exterior power is a direct summand of the tensor power [9, III, §7.4], this implies the claim. □

Proposition 3.14. *Let $X \subseteq \mathbb{P}^n$ be a smooth irreducible subvariety of codimension e and degree at most $D \geq 2$. Then*

$$\mathrm{reg}\,(\Omega_X) \leq (e+1)D - e.$$

Proof. Denote with $\mathcal{I}$ the ideal sheaf of X and let $i \colon X \hookrightarrow \mathbb{P}^n$ be the inclusion. Part (ii) of Remark 3.2 applied to the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_X \to 0$$

implies $\mathrm{reg}\,(\mathcal{O}_X) = \mathrm{reg}\,(i_*\mathcal{O}_X) \leq \max\{\mathrm{reg}\,(\mathcal{I}) - 1, \mathrm{reg}\,(\mathcal{O}_{\mathbb{P}^n})\}$. Using Corollary 3.9 and Example 3.3 (i) we conclude

$$\mathrm{reg}\,(\mathcal{O}_X) \leq eD - e. \tag{6}$$

Furthermore, from the exact sequence $0 \to \mathcal{I}^2 \to \mathcal{I} \to \mathcal{I}/\mathcal{I}^2 \to 0$ it follows

$$\mathrm{reg}\,(\mathcal{I}/\mathcal{I}^2) \leq \max\{\mathrm{reg}\,(\mathcal{I}), \mathrm{reg}\,(\mathcal{I}^2) - 1\} \leq (e+1)D - e. \tag{7}$$

The last exact sequence we consider is the conormal sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \mathcal{O}_X \otimes \Omega_{\mathbb{P}^n} \to \Omega_X \to 0 \tag{8}$$

on X [32, II, Theorem 8.17]. More precisely, the sheaf in the middle is the inverse image

$$i^*\Omega_{\mathbb{P}^n} = \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_{\mathbb{P}^n}} i^{-1}\Omega_{\mathbb{P}^n}$$

of $\Omega_{\mathbb{P}^n}$ under i as an $\mathcal{O}_X$-module. The sheaf on the left is the restriction $i^{-1}\mathcal{I}/\mathcal{I}^2$, which is automatically an $\mathcal{O}_X$-module.

We want to push the sequence (8) forward to $\mathbb{P}^n$. In general, the direct image of sheaves does not commute with stalks. However, for closed immersions it does: if $\mathcal{F}$ is a sheaf on X, we have $(i_*\mathcal{F})_x = \mathcal{F}_x$ for $x \in X$, and 0 otherwise [32, II, Exercise 1.19(a)]. It follows that the functor i_* is exact.

Furthermore, since $\Omega_{\mathbb{P}^n}$ is locally free of finite rank, the projection formula shows $i_*i^*\Omega_{\mathbb{P}^n} = i_*\mathcal{O}_X \otimes_{\mathcal{O}_{\mathbb{P}^n}} \Omega_{\mathbb{P}^n}$. Since the stalk of $\mathcal{I}/\mathcal{I}^2$ at $x \notin X$ vanishes, we have $i_*i^{-1}\mathcal{I}/\mathcal{I}^2 = \mathcal{I}/\mathcal{I}^2$. Thus, applying i_* to (8) yields the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i_*\mathcal{O}_X \otimes \Omega_{\mathbb{P}^n} \to i_*\Omega_X \to 0.$$

Hence, $\operatorname{reg}(\Omega_X) = \operatorname{reg}(i_*\Omega_X) \leq \max\{\operatorname{reg}(i_*\mathcal{O}_X \otimes \Omega_{\mathbb{P}^n}), \operatorname{reg}(\mathcal{I}/\mathcal{I}^2)\}$. Now Proposition 3.12, Example 3.3 (iii), and (6) imply $\operatorname{reg}(i_*\mathcal{O}_X \otimes \Omega_{\mathbb{P}^n}) \leq \operatorname{reg}(\mathcal{O}_X) + \operatorname{reg}(\Omega_{\mathbb{P}^n}) \leq eD - e + 2$, so that $\operatorname{reg}(\Omega_X) \leq (e+1)D - e$ by (7). □

Remark 3.15. (i) In the case $D = 1$ we have $\operatorname{reg}(\Omega_X) \leq 2$.

(ii) One can check that for a hypersurface $X \subseteq \mathbb{P}^n$ of degree $D \geq 3$ one has $\operatorname{reg}(\Omega_X) = 2D - 2$, so that our bound is essentially sharp.

Proof of Theorem 3.4. The claim can easily be checked for $D = 1$. Also, we can assume X to be irreducible. The claim for $p = 0$ is (6). For the case $p \geq 1$, Proposition 3.14 and Corollary 3.13 imply

$$\begin{aligned}\operatorname{reg}(\Omega_X^p) &\leq p((e+1)D - e) + (p-1)(m-1)(e(D-1)-1) \\ &< p\,((e+1)D - e + (m-1)e(D-1)) \\ &< p(em+1)D.\end{aligned}$$

□

4 Cohomology of Hypersurface Complements

4.1 Theory

Let $X \subseteq \mathbb{P}^n$ be a smooth closed subvariety. Using our result on the Castelnuovo-Mumford regularity of the sheaf of differential forms, one can compute the de Rham cohomology of certain hypersurface complements in X as the cohomology of finite dimensional complexes.

To describe these complexes, let $H_0, \ldots, H_q \subseteq X$ be hyperplane sections and denote by U the complement of the hypersurface $V := \bigcup_\nu H_\nu$ in X. Assume that V has normal crossings (see §2.3). We also consider V as a divisor $V = \sum_\nu H_\nu = \sum_i m_i V_i$, where the V_i are the irreducible components of V (cf. §2.3). Then, since $\mathcal{O}_X(H_\nu) \simeq \mathcal{O}_X(1)$, it follows that

$$\mathcal{O}_X(V) \simeq \bigotimes_\nu \mathcal{O}(H_\nu) \simeq \mathcal{O}_X(1)^{\otimes(q+1)} \simeq \mathcal{O}_X(q+1). \tag{9}$$

Now let $A = \sum_i a_i V_i$ be any divisor with support in V, and let $j\colon U \hookrightarrow X$ be the inclusion. Define the subsheaf $\Omega_X^p(A) := \Omega_X^p \otimes \mathcal{O}_X(A)$ of $j_*\Omega_U^p$, which consists of those rational differential p-forms on X, which are regular on U and have poles (zeros if $a_i < 0$) of order $|a_i|$ along V_i. Define the sheaves

$$\mathcal{K}_X^p(A) := \Omega_X^p(A + pV).$$

Note that $\mathrm{d}(\mathcal{K}_X^p(A)) \subseteq \mathcal{K}_X^{p+1}(A)$, so that $\mathcal{K}_X^\bullet(A)$ is in fact a subcomplex of $j_*\Omega_U^\bullet$. For $A = V$ it is the zeroth term of the *polar filtration* [16, 17].

The next lemma is the crucial fact that allows us to compute the algebraic de Rham cohomology of U by a finite dimensional complex. Its proof requires to consider holomorphic differential forms. So let $\Omega_{U^{\mathrm{an}}}^\bullet$ denote the complex of holomorphic differential forms on U^{an} regarded as a complex manifold, and let $\mathcal{K}_{X^{\mathrm{an}}}^\bullet(A)$ be the holomorphic version of $\mathcal{K}_X^\bullet(A)$. The following lemma is proved analogously to the corresponding statement for the logarithmic complex (cf. [15, 29, 59]). The calculation can be found in [2, Lemma 4.1].

Lemma 4.1. *Let $A = \sum_i a_i V_i$ be a divisor with $a_i > 0$ for all i, and assume that V has normal crossings. Then the inclusion $\mathcal{K}_{X^{\mathrm{an}}}^\bullet(A) \hookrightarrow j_*\Omega_{U^{\mathrm{an}}}^\bullet$ is a quasi-isomorphism.*

The following is the main result of this section and the key for our algorithm.

Theorem 4.2. *Let $X \subseteq \mathbb{P}^n$ be a smooth closed subvariety with dimension at most $m \geq 1$. Let $D \geq 2$ and e be upper bounds on the degree and the codimension of all irreducible components of X. Let $H_0, \ldots, H_q$ be hyperplane sections of X such that $V = H_0 \cup \ldots \cup H_q$ has normal crossings, and denote $U := X \setminus V$. For $s \in \mathbb{N}$ set $K_s^\bullet := \Gamma(X, \mathcal{K}_X^\bullet(sV))$. Then we have*

$$H_{\mathrm{dR}}^\bullet(U) \simeq H^\bullet(K_s^\bullet) \quad \text{for} \quad s \geq m(em+1)D.$$

Proof. It follows from Lemma 4.1 that

$$\mathbb{H}^\bullet(X^{\mathrm{an}}, \mathcal{K}_{X^{\mathrm{an}}}^\bullet(sV)) \simeq \mathbb{H}^\bullet(X^{\mathrm{an}}, j_*\Omega_{U^{\mathrm{an}}}^\bullet). \tag{10}$$

Since $\mathcal{K}_X^\bullet(sV)$ and $j_*\Omega_U^\bullet$ are coherent sheaves on X, by GAGA [53] the hypercohomologies in (10) can be replaced by their algebraic versions. But we have $H^i(X, j_*\Omega_U^p) = H^i(U, \Omega_U^p) = 0$ for $i > 0$ since U is affine (see §2.2). Hence, the right hypercohomology in (10) is $H_{\mathrm{dR}}^\bullet(U)$.

On the other hand, Theorem 3.4 implies $s \geq \operatorname{reg}(\Omega_X^p)$ for all $0 \leq p \leq m$, thus

$$H^i(X, \Omega_X^p((s+p)V)) = H^i(X, \Omega_X^p((s+p)(q+1))) = 0 \quad \text{for all} \quad i > 0$$

(use (9)). It follows that the left side of (10) is $H^\bullet(K_s^\bullet)$ as claimed. □

4.2 Computation

We adopt the notations and assumptions of the last section. We choose s according to Theorem 4.2 and set $K^\bullet := K_s^\bullet$. In this section we describe this finite dimensional complex more explicitly and show how to compute its cohomology.

Let $H_\nu = \mathcal{Z}_X(\ell_\nu)$ with linear forms ℓ_ν, $0 \le \nu \le q$. We can assume w.l.o.g. that $\ell_0 = X_0$. With $f := X_0\ell_1 \cdots \ell_q$ we have $V = \mathcal{Z}_X(f)$ and $U = X \setminus V$. On the ambient space we define $\widetilde{V} := \mathcal{Z}(f)$ and $\widetilde{U} := \mathbb{P}^n \setminus \widetilde{V}$. Recall from §2.2 that each p-form on $\widetilde{U}$ is given by

$$\omega = \frac{\alpha}{f^t} \quad \text{with} \quad \deg \alpha = t(q+1) \quad \text{and} \quad \Delta(\alpha) = 0, \tag{11}$$

where α is a homogeneous p-form on $\mathbb{C}^{n+1}$ and Δ denotes contraction with the Euler vector field. Note that for fixed t such forms are precisely the global sections of the sheaf $\Omega^p_{\mathbb{P}^n}(t\widetilde{V})$. We have the following

Lemma 4.3. *With $s \ge m(em+1)D$ the restriction map*

$$\Gamma(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(t\widetilde{V})) \longrightarrow \Gamma(X, \Omega^p_X(tV))$$

is surjective for all $t \ge s$ and $p \ge 0$.

Proof. Let $i\colon X \hookrightarrow \mathbb{P}^n$ be the inclusion and consider the exact sequence

$$0 \to \mathcal{I} \otimes \Omega^p_{\mathbb{P}^n} \to \Omega^p_{\mathbb{P}^n} \xrightarrow{\alpha} i_*\mathcal{O}_X \otimes \Omega^p_{\mathbb{P}^n} \to 0 \tag{12}$$

on $\mathbb{P}^n$, as well as

$$0 \to \ker\beta \to i^*\Omega^p_{\mathbb{P}^n} \xrightarrow{\beta} \Omega^p_X \to 0 \tag{13}$$

on X. The restriction map from the lemma coincides with the composition of

$$H^0(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(t(q{+}1))) \xrightarrow{\alpha^*} H^0(\mathbb{P}^n, i_*\mathcal{O}_X \otimes \Omega^p_{\mathbb{P}^n}(t(q{+}1))) \simeq H^0(X, i^*\Omega^p_{\mathbb{P}^n}(t(q{+}1)))$$

with

$$H^0(X, i^*\Omega^p_{\mathbb{P}^n}(t(q+1))) \xrightarrow{\beta^*} H^0(X, \Omega^p_X(t(q+1))).$$

We show that α^* and β^* are surjective for $t \ge s$. For α^* this follows immediately by twisting the short exact sequence (12) and considering the induced long exact sequence of cohomology, taking into account that $H^1(\mathbb{P}^n, \mathcal{I} \otimes \Omega^p_{\mathbb{P}^n}(t(q{+}1))) = 0$, since

$$\operatorname{reg}(\mathcal{I} \otimes \Omega^p_{\mathbb{P}^n}) \le \operatorname{reg}(\mathcal{I}) + p \cdot \operatorname{reg}(\Omega_{\mathbb{P}^n}) \le e(D-1) + 1 + 2p \le s + 1.$$

We can prove that β^* is surjective by the same method, once we bound the regularity of $\ker\beta$. By [32, II, Exercise 5.16] the exact sequence

$$0 \to \mathcal{I}/\mathcal{I}^2 \to i^*\Omega_{\mathbb{P}^n} \to \Omega_X \to 0$$

induces a filtration of locally free sheaves on X

$$\bigwedge^p (i^*\Omega_{\mathbb{P}^n}) = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^p \supseteq \mathcal{F}^{p+1} = 0$$

and for all $0 \le j \le p$ an exact sequence

$$0 \to \mathcal{F}^{j+1} \to \mathcal{F}^j \to \bigwedge^j \mathcal{I}/\mathcal{I}^2 \otimes \Omega^{p-j}_X \to 0 \tag{14}$$

In particular, for $j=0$ this sequence coincides with (13), since $\bigwedge^p(i^*\Omega_{\mathbb{P}^n}) = i^*\Omega^p_{\mathbb{P}^n}$ [32, II, Exercise 5.16], so that $\ker\beta \simeq \mathcal{F}^1$.

The same calculation as in the proof of Theorem 3.4 using Proposition 3.14 and Corollary 3.13 shows

$$\operatorname{reg}\big(\bigwedge^j \mathcal{I}/\mathcal{I}^2 \otimes \Omega_X^{p-j}\big) \le p(em+1)D \le s.$$

Furthermore, by Remark 3.2 (ii) the sequence (14) yields

$$\operatorname{reg}(\mathcal{F}^j) \le \max\{\operatorname{reg}(\mathcal{F}^{j+1}), s\},$$

which inductively implies $\operatorname{reg}(\mathcal{F}^j) \le s$ for all j, in particular $\operatorname{reg}(\ker\beta) \le s$. □

It follows from the Lemma that each element of $K^p = \Gamma(X, \Omega_X^p((s+p)V))$ is the restriction of a form in $\Omega^p := \Gamma(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(t\widetilde{V}))$.

We identify $\mathbb{C}^n \simeq \{X_0 \neq 0\} \subseteq \mathbb{P}^n$ and set $X^0 := X \setminus \mathcal{Z}(X_0)$. As with polynomials one can dehomogenize a homogeneous differential form α on $\mathbb{C}^{n+1}$ by setting $X_0 = 1$ and $\mathrm{d}X_0 = 0$ to get a form α^0 on $\mathbb{C}^n$. Hence for $\omega \in \Omega^p$ one gets a regular form ω^0 on $\mathbb{C}^n \setminus \mathcal{Z}(f^0)$. Its restriction defines a regular form on the dense open subset $U = X^0 \setminus \mathcal{Z}(f^0)$ of X^0.

We use the algorithm of Szántó [56, 57] to compute a decomposition $I := I(X^0) = \bigcap_j I_j$, where each I_j is the saturated ideal of a squarefree regular chain G_j. Note that $\mathcal{Z}(I_j)$ is equidimensional. We will construct for all j a linear system of equations describing the identity $\omega = 0$ on $\mathcal{Z}(I_j)$ for $\omega \in K^p$. For simplicity we assume that I is represented by a single $G = \{g_1, \ldots, g_e\}$. In the general case one only has to combine all the linear systems to one large system.

Let $k \in \mathbb{N}$. In [12] we have constructed a linear system of equations

$$\operatorname{prem}_k(f, G) = 0 \tag{15}$$

in the coefficients of $f \in \mathbb{C}[X_1, \ldots, X_n]$, whose solution space is $I_{\le k}$, the set of polynomials of degree $\le k$ vanishing on X^0.

Szántó's algorithm also yields a polynomial h which is a non-zerodivisor mod I, and such that the module of differentials on $X^0 \setminus \mathcal{Z}(h)$ is the free module generated by m of the $\mathrm{d}X_j$, where $m = \dim X^0$. More precisely, let $X_1, \ldots, X_m$ denote the free variables, and $Y_1, \ldots, Y_e$ the dependent variables, where $m+e=n$. For a polynomial $F \in \mathbb{C}[X_1, \ldots, X_m, Y_1, \ldots, Y_e]$ we denote $\overline{F} := F \bmod I \in \mathbb{C}[X^0]$. Then by [12, Propositon 3.13] we have $\Omega_{\mathbb{C}[X^0]_h/\mathbb{C}} = \bigoplus_{i=1}^m \mathbb{C}[X^0]_h \mathrm{d}\overline{X}_i$. Furthermore, for all $F \in \mathbb{C}[X_1, \ldots, X_m, Y_1, \ldots, Y_e]$

$$\mathrm{d}\overline{F} = \sum_{i=1}^m \left(\frac{\partial F}{\partial X_i} - \frac{\partial F}{\partial Y} \left(\frac{\partial g}{\partial Y}\right)^{-1} \frac{\partial g}{\partial X_i} \right) \mathrm{d}\overline{X}_i, \tag{16}$$

where $g := (g_1, \ldots, g_e)^T$. Note that h is a multiple of $\det(\frac{\partial g}{\partial Y})$, so that the entries of $(\frac{\partial g}{\partial Y})^{-1}$ lie in $h^{-1}\mathbb{C}[X_1, \ldots, X_n]$. Using (16) for the coordinates Y_j, one can write the restriction of a form $\omega \in \Omega^p$ to $U \setminus \mathcal{Z}(h)$ in terms of the free generators of $\Omega^p_{\mathbb{C}[U]_h/\mathbb{C}}$, which are $\mathrm{d}\overline{X}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{X}_{i_p}$, where $1 \leq i_1 < \cdots < i_p \leq m$. It follows that $\omega = \frac{\omega_h}{h(f^0)^t}$, where

$$\omega_h = \sum_{1 \leq i_1 < \cdots < i_p \leq m} (\omega_h)_{i_1 \cdots i_p} \mathrm{d}\overline{X}_{i_1} \wedge \cdots \wedge \mathrm{d}\overline{X}_{i_p} \in \Omega^p_{\mathbb{C}[X^0]/\mathbb{C}}.$$

Then, since $U \setminus \mathcal{Z}(h)$ is dense in U and in X^0, we have

$$\begin{aligned} \omega = 0 \text{ on } U \quad & \Longleftrightarrow \quad \omega_h = 0 \text{ on } U \setminus \mathcal{Z}(h) \\ & \Longleftrightarrow \quad \forall i_1 < \cdots < i_p \colon \; (\omega_h)_{i_1 \cdots i_p} = 0 \text{ on } U \setminus \mathcal{Z}(h) \\ & \Longleftrightarrow \quad \forall i_1 < \cdots < i_p \colon \; (\omega_h)_{i_1 \cdots i_p} \in I_{\leq k} \end{aligned} \tag{17}$$

for sufficiently large k.

Now we compute the cohomology of $K^\bullet$. First note that the contraction with the Euler vector field Δ (cf. §2.2) can be easily computed, so that we can compute a basis for Ω^p. Consider the commutative diagram

$$\begin{array}{ccc} \Omega^p & \xrightarrow{\mathrm{d}} & \Omega^{p+1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ K^p & \xrightarrow{\mathrm{d}} & K^{p+1}, \end{array}$$

where π is the restriction of forms to U. Let $N^p := \ker(\pi \colon \Omega^p \to K^p)$. According to (17) and (15), N^p is the solution set of a linear system of equations. Since π is surjective, we have $K^p \simeq \Omega^p / N^p$, thus K^p can be identified with any complementary subspace of N^p in Ω^p. So we compute a basis of N^p and extend it to a basis of Ω^p to get a basis of K^p via this identification. The differential $\mathrm{d} \colon K^p \to K^{p+1}$ is just the restriction of the differential $\mathrm{d} \colon \Omega^p \to \Omega^{p+1}$, which we can evaluate efficiently. Hence we can compute the matrix of $\mathrm{d} \colon K^p \to K^{p+1}$ with respect to the computed bases of $K^\bullet$. By computing kernel and image of this matrix and taking their quotient we get the cohomology of $K^\bullet$.

Proposition 4.4. *Under the notations and assumptions of Theorem 4.2, let $X \subseteq \mathbb{P}^n$ be given by equations of degree $\leq d$. Then one can compute the cohomology $H^\bullet_{\mathrm{dR}}(U)$ in parallel time $(d \log n)^{\mathcal{O}(1)}$ and sequential time $d^{\mathcal{O}(n^4)}$.*

Proof. It remains to analyze the algorithm described above. Let δ denote the maximal degree of the polynomials in the squarefree regular chain G. Then the system (15) has asymptotic size $\mathcal{O}((nk\delta^e)^n)$ and can be computed in parallel time $(n \log(k\delta))^{\mathcal{O}(1)}$ and sequential time $(k\delta)^{\mathcal{O}(n^4)}$ [12]. Since the numerator of each $\omega \in \Omega^p$ has degree $(s+p)(q+1)$, the dimension of Ω^p is $\binom{n+1}{p}\binom{(s+p)(q+1)-p+n}{n} = \mathcal{O}(s^n n^{\mathcal{O}(n)})$. Furthermore, for $\omega \in \Omega^p$, the degree of the coefficients of ω_h is bounded by $(s+p)(q+1) + p(e+1)\delta$, hence we must choose the k in (17) of that order. Thus, N^p is described by a linear system of equations of size $\mathcal{O}(n^n((s+n)n+n^2\delta)^n\delta^{en}) \leq n^{\mathcal{O}(n)}s^n\delta^{(e+1)n}$. Now let X be given by equations of degree d. According to Theorem 4.2, we have to choose s of order $n^3 \deg X \leq n^3 d^n$. Furthermore, by [57] we have $\delta = d^{\mathcal{O}(n^2)}$. Hence the size of this system is $d^{\mathcal{O}(n^4)}$. The algorithms of §2.7 imply the claimed bounds. □

5 Patching Cohomologies

Let X be a smooth projective variety of dimension at most $m \geq 1$. Our aim is to compute the de Rham cohomology of X by way of an open affine cover.

So let $H_0, \ldots, H_m \subseteq X$ be hyperplane sections with $H_0 \cap \cdots \cap H_m = \emptyset$ and set $U_i := X \setminus H_i$. Then $\mathcal{U} := \{U_i \mid 0 \leq i \leq m\}$ is an open affine cover of X. Consider the *Čech double complex* $C^{\bullet,\bullet} := C^{\bullet,\bullet}(\mathcal{U}, \Omega_X^\bullet)$ as defined in §2.5. Recall that with $U_{i_0\cdots i_q} = U_{i_0} \cap \cdots \cap U_{i_q}$ we have $C^{p,q}(\mathcal{U}, \Omega_X^\bullet) = \bigoplus_{i_0<\cdots<i_q} \Omega_X^p(U_{i_0\cdots i_q})$. Since $\mathcal{U}$ is a Leray cover for all the sheaves Ω_X^p, we have

Lemma 5.1. $H_{\mathrm{dR}}^\bullet(X) \simeq \mathbb{H}^\bullet(\mathcal{U}, \Omega_X^\bullet) = H^\bullet(\mathrm{tot}^\bullet(C^{\bullet,\bullet}))$, *where* $\mathrm{tot}^\bullet(C^{\bullet,\bullet})$ *denotes the total complex associated to* $C^{\bullet,\bullet}$.

To compute this cohomology, we replace the infinite dimensional double complex $C^{\bullet,\bullet}$ by a finite dimensional one, which is built from the complex of the last section for each $U_{i_0\cdots i_q}$. More precisely, let e, D have the meanings of Theorem 4.2, and choose $s \geq m(em+1)D$. For a hypersurface V in X we denote $K^p(V) := \Gamma(X, \Omega_X^p((s+p)V))$. This corresponds to the complex $K_s^\bullet$ from Theorem 4.2. Now we define the double complex

$$K^{p,q} := \bigoplus_{i_0<\cdots<i_q} K^p(H_{i_0} \cup \cdots \cup H_{i_q})$$

together with the differential $\delta\colon K^{p,q} \to K^{p,q+1}$, which is the restriction of the Čech differential (3), and the exterior differential $\mathrm{d}\colon K^{p,q} \to K^{p+1,q}$. Then $K^{\bullet,\bullet}$ is a subcomplex of $C^{\bullet,\bullet}$.

Lemma 5.2. *We have* $H_{\mathrm{dR}}^\bullet(X) \simeq H^\bullet(\mathrm{tot}^\bullet(K^{\bullet,\bullet}))$.

Proof. Clearly, the inclusion $K^{\bullet,\bullet} \hookrightarrow C^{\bullet,\bullet}$ induces a morphism of spectral sequences $''E_r(K^{\bullet,\bullet}) \to {''E_r}(C^{\bullet,\bullet})$ between the second spectral sequences of these double complexes. Theorem 4.2 implies that this is an isomorphism

$$''E_1^{p,q}(K^{\bullet,\bullet}) \simeq \bigoplus_{i_0<\cdots<i_q} H_{\mathrm{dR}}^p(U_{i_0\cdots i_q}) = {''E_1^{p,q}}(C^{\bullet,\bullet})$$

on the first level of these spectral sequences. According to [41, Theorem 3.5], this induces an isomorphism on their ∞-terms and, since the corresponding filtrations are bounded, also on the cohomologies of the total complexes, so $H^\bullet(\mathrm{tot}^\bullet(K^{\bullet,\bullet})) \simeq H^\bullet(\mathrm{tot}^\bullet(C^{\bullet,\bullet}))$. Together with Lemma 5.1 this completes the proof. □

Proposition 5.3. *Assume that one is given homogeneous polynomials of degree at most d defining the smooth variety $X \subseteq \mathbb{P}^n$, and linear forms defining the hyperplane sections $H_0, \ldots, H_m$ such that $\bigcup_i H_i$ has normal crossings. Then one can compute $H^\bullet_{\mathrm{dR}}(X)$ in parallel time $(d \log n)^{\mathcal{O}(1)}$ and sequential time $d^{\mathcal{O}(n^4)}$.*

Proof. By Lemma 5.2 one has to compute the cohomology of the total complex $T^k := \mathrm{tot}^k(K^{\bullet,\bullet}) = \bigoplus_{p+q=k} K^{p,q}$ with the differential

$$d_T \colon T^k \to T^{k+1},\ (\omega_{p,q})_{p+q=k} \mapsto (\mathrm{d}\omega_{p-1,q} + (-1)^p \delta\omega_{p,q-1})_{p+q=k+1}\,.$$

As in §4.2 one can compute bases for $K^{p,q}$ and hence for $T^\bullet$ within the claimed bounds. Since the differential d_T is easily computable and the vector spaces have dimension $d^{\mathcal{O}(n^4)}$, the cohomology of this complex can be computed using the algorithms of §2.7. □

We conclude this section with an

Example 5.4. Consider the plane curve $X = \mathcal{Z}(X_0^3 + X_1^3 + X_2^3) \subseteq \mathbb{P}^2$. One easily checks that X is smooth, and the well known genus formula shows that its Betti numbers are

$$b_0(X) = b_2(X) = 1, \quad b_1(X) = 2.$$

Furthermore, the hyperplanes $\widetilde{H}_0 := \mathcal{Z}(X_0)$ and $\widetilde{H}_1 := \mathcal{Z}(X_1)$ intersect X transversally, and their hyperplane sections $H_0 := \mathcal{Z}_X(X_0)$ and $H_1 := \mathcal{Z}_X(X_1)$ do not intersect. Hence, $U_i := X \setminus H_i$, $i \in \{0,1\}$, form an open cover of X satisfying our assumptions.

Our theorems yield the lower bound $s \geq 6$, but let us find the smallest possible s. The bounds of Theorem 3.4 are $\mathrm{reg}\,(\mathcal{O}_X) \leq 2$ and $\mathrm{reg}\,(\Omega_X) \leq 6$, but the more precise Proposition 3.14 yields $\mathrm{reg}\,(\Omega_X) \leq 5$. However, computations with Macaulay2 [28] give $H^1(\mathcal{O}_X(1)) = H^1(\Omega_X(1)) = 0$ and $H^1(\mathcal{O}_X) \neq 0$, $H^1(\Omega_X) \neq 0$, hence $\mathrm{reg}\,(\mathcal{O}_X) = \mathrm{reg}\,(\Omega_X) = 2$ (since $\dim X = 1$). A close look at the proofs shows that the conclusions of Theorem 4.2 and Lemma 5.2 hold for $s = 1$. But Macaulay2 also computes $\dim H^0(\Omega_{\mathbb{P}^2}(2)) = 3$ and $\dim H^0(\Omega_X(2)) = 6$, so the restriction map in Lemma 4.3 cannot be surjective. In order to get surjectivity, note that $\mathrm{reg}\,(\mathcal{I}) = 3$, thus the restriction map $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t\widetilde{H}_0)) \to \Gamma(X, \mathcal{O}_X(tH_0))$ is surjective for $t \geq 2$. As for the one-forms, we compute $\mathrm{reg}\,(\mathcal{I} \otimes \Omega_{\mathbb{P}^2}) = \mathrm{reg}\,(\mathcal{I}/\mathcal{I}^2) = 5$. It follows that the corresponding restriction map is surjective for $t \geq 4$, so we can choose $s = 3$.

The double complex we have to consider is

$$\begin{array}{ccc} K^{0,1} & \xrightarrow{\mathrm{d}^1} & K^{1,1} \\ {\scriptstyle \delta^0}\uparrow & & {\scriptstyle \delta^1}\uparrow \\ K^{0,0} & \xrightarrow{\mathrm{d}^0} & K^{1,0}, \end{array}$$

where

$$K^{0,0} = K^0(H_0) \oplus K^0(H_1) = \Gamma(X, \mathcal{O}_X(3H_0)) \oplus \Gamma(X, \mathcal{O}_X(3H_1)),$$

$$
\begin{aligned}
K^{1,0} = K^1(H_0) \oplus K^1(H_1) = \Gamma(X, \Omega_X(4H_0)) \oplus \Gamma(X, \Omega_X(4H_1)),\\
K^{0,1} = K^0(H_0 \cup H_1) = \Gamma(X, \mathcal{O}_X(3H_0 + 3H_1)),\\
K^{1,1} = K^1(H_0 \cup H_1) = \Gamma(X, \Omega_X(4H_0 + 4H_1)),
\end{aligned}
$$

together with the differentials

$$\delta^0(f,g) = g - f, \quad \mathrm{d}^0(f,g) = (\mathrm{d}f, \mathrm{d}g), \quad \delta^1(\omega,\eta) = \eta - \omega, \quad \mathrm{d}^1(f) = \mathrm{d}f.$$

The dimensions of these vector spaces are

$$\dim K^{0,0} = \dim K^{0,1} = 18, \quad \dim K^{1,0} = \dim K^{1,1} = 24.$$

Put $\Omega_{ij} = X_i \mathrm{d}X_j - X_j \mathrm{d}X_i$ (for simplicity we write X_i instead of $\overline{X}_i$). Then, bases for these spaces are given by

$$K^0(H_0): \quad \frac{1}{X_0^3}(X_0^3, X_0^2X_1, X_0^2X_2, X_0X_1^2, X_0X_1X_2, X_0X_2^2, X_1^3, X_1^2X_2, X_1X_2^2)$$

$$K^0(H_1): \quad \frac{1}{X_1^3}(X_0^3, X_0^2X_1, X_0^2X_2, X_0X_1^2, X_0X_1X_2, X_0X_2^2, X_1^3, X_1^2X_2, X_1X_2^2)$$

$$
\begin{aligned}
K^1(H_0): \quad & \frac{1}{X_0^4}(X_0^2\Omega_{01}, X_0X_1\Omega_{01}, X_0X_2\Omega_{01}, X_1^2\Omega_{01}, X_1X_2\Omega_{01}, X_2^2\Omega_{01},\\
& \quad X_0^2\Omega_{02}, X_0X_1\Omega_{02}, X_0X_2\Omega_{02}, X_1^2\Omega_{02}, X_1X_2\Omega_{02}, X_1X_2\Omega_{12})
\end{aligned}
$$

$$
\begin{aligned}
K^1(H_1): \quad & \frac{1}{X_1^4}(X_0^2\Omega_{01}, X_0X_1\Omega_{01}, X_0X_2\Omega_{01}, X_1^2\Omega_{01}, X_1X_2\Omega_{01}, X_2^2\Omega_{01},\\
& \quad X_0^2\Omega_{02}, X_0X_1\Omega_{02}, X_0X_2\Omega_{02}, X_1^2\Omega_{02}, X_1X_2\Omega_{02}, X_1X_2\Omega_{12})
\end{aligned}
$$

$K^0(H_0 \cup H_1)$:

$$
\begin{aligned}
& \frac{1}{(X_0X_1)^3}(X_0^6, X_0^5X_1, X_0^5X_2, X_0^4X_1^2, X_0^4X_1X_2, X_0^4X_2^2, X_0^3X_1^3, X_0^3X_1^2X_2,\\
& \quad X_0^3X_1X_2^2, X_0^2X_1^4, X_0^2X_1^3X_2, X_0^2X_1^2X_2^2, X_0X_1^5, X_0X_1^4X_2,\\
& \quad X_0X_1^3X_2^2, X_1^6, X_1^5X_2, X_1^4X_2^2)
\end{aligned}
$$

$K^1(H_0 \cup H_1)$:

$$
\begin{aligned}
& \frac{\Omega_{01}}{(X_0X_1)^4}(X_0^6, X_0^5X_1, X_0^5X_2, X_0^4X_1^2, X_0^4X_1X_2, X_0^4X_2^2, X_0^3X_1^3, X_0^3X_1^2X_2,\\
& \quad X_0^3X_1X_2^2, X_0^2X_1^4, X_0^2X_1^3X_2, X_0^2X_1^2X_2^2, X_0X_1^5, X_0X_1^4X_2,\\
& \quad X_0X_1^3X_2^2, X_1^6, X_1^5X_2, X_1^4X_2^2),\\
& \frac{\Omega_{02}}{(X_0X_1)^4}(X_0^6, X_0^5X_1, X_0^5X_2, X_0^4X_1^2, X_0^4X_1X_2, X_0^3X_1^2X_2)
\end{aligned}
$$

The corresponding total complex $T^\bullet = \mathrm{tot}^\bullet(K^{\bullet,\bullet})$ is

$$T^0 = K^{0,0} \xrightarrow{\mathrm{d}^{\mathrm{tot},0}} T^1 = K^{0,1} \oplus K^{1,0} \xrightarrow{\mathrm{d}^{\mathrm{tot},1}} T^2 = K^{1,1},$$

where

$$\mathrm{d}^{\mathrm{tot},0} = (\delta^0, \mathrm{d}^0), \quad \mathrm{d}^{\mathrm{tot},1} = \mathrm{d}^1 - \delta^1.$$

The matrix of $\mathrm{d}^{\mathrm{tot},0}$ with respect to the given bases is

$$\left(\begin{array}{cccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right).$$

The matrix of $\mathrm{d}^{\mathrm{tot},1}$ is the transpose of

$$\left(\begin{array}{cccccccccccccccccccccccc}
-3&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2\\
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-2&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&-2&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&3&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&-1\\
\hline
0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&-1\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1
\end{array}\right).$$

Since the rank of $\mathrm{d}^{\mathrm{tot},1}$ is 23, $H^2_{\mathrm{dR}}(X) = H^2(T^\bullet)$ has the right dimension, and since the seventh column of its matrix is zero, the corresponding basis vector is not in the image, hence $H^2_{\mathrm{dR}}(X)$ is spanned by

$$\frac{\Omega_{01}}{X_0X_1} = \frac{\mathrm{d}X_1}{X_1} - \frac{\mathrm{d}X_0}{X_0} \in K^1(H_0 \cup H_1).$$

The nullity of $\mathrm{d}^{\mathrm{tot},1}$ is 19, and a basis for its kernel is given by the columns of

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&-1&0&0&-1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&1&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0\\
0&0&-1&0&0&-1&0&0&0&0&0&0&0&-1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0\\
0&0&2&0&0&2&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&3&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&3&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&-2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}.$$

The rank of $\mathrm{d}^{\mathrm{tot},0}$ is 17, and by some rank computations one sees that the third and fourth column of the above matrix are a basis of the first cohomology $H^1_{\mathrm{dR}}(X) = H^1(T^\bullet)$, which correspond to

$$\left(-\frac{X_1X_2}{X_0^2}, 2\frac{X_2\Omega_{01}}{X_0^3}, \frac{\Omega_{02}}{X_1^2}\right), \left(-\frac{X_2^2}{X_0X_1}, -\frac{X_1X_2\Omega_{12}}{X_0^4}, \frac{X_0X_2\Omega_{02}}{X_1^4}\right)$$

$$\in K^0(H_0 \cup H_1) \oplus K^1(H_0) \oplus K^1(H_1).$$

Finally, the nullity of $\mathrm{d}^{\mathrm{tot},0}$ is 1, and the third last column cancels with the first, so $H^0_{\mathrm{dR}}(X) = H^0(T^\bullet)$ is generated by

$$(1, 1) \in K^0(H_0) \oplus K^0(H_1).$$

Now let us see if we can find the Hodge decomposition

$$H^1_{\mathrm{dR}}(X) \simeq H^1(\mathcal{O}_X) \oplus H^0(\Omega_X)$$

in our double complex $K^{\bullet,\bullet}$. For that purpose, note that δ^0 is the submatrix of $\mathrm{d}^{\mathrm{tot},1}$ above the line, and δ^1 is the transpose of the submatrix of $\mathrm{d}^{\mathrm{tot},1}$ below the line. Since the 12^{th} row of δ^0 is zero, $H^1(\mathcal{O}_X)$ is spanned by

$$\frac{X_2^2}{X_0X_1} \in K^0(H_0 \cup H_1).$$

Furthermore, one finds that the kernel of δ^1 and hence $H^0(\Omega_X)$ is generated by

$$\left(\frac{X_2\Omega_{01} - X_1\Omega_{02}}{X_3^3}, \frac{\Omega_{02}}{X_1^2}\right) = \left(\frac{-\Omega_{12}}{X_0^2}, \frac{\Omega_{02}}{X_1^2}\right) \in K^1(H_0) \oplus K^1(H_1).$$

So indeed, we find generaters for the sheaf cohomologies of $\mathcal{O}_X$ and Ω_X in our vertical subcomplexes $K^{p,\bullet}$. It is interesting to note that $\left(0, \frac{-\Omega_{12}}{X_0^2}, \frac{\Omega_{02}}{X_1^2}\right)$ can also be chosen as one vector in a basis of $H^1_{\mathrm{dR}}(X)$, but $(\frac{X_2^2}{X_0X_1}, 0, 0)$ cannot. In fact, it is easy to see that any element of the form $(f, 0, 0)$ in the kernel of $\mathrm{d}^{\mathrm{tot},1}$ is in the image of $\mathrm{d}^{\mathrm{tot},0}$.

6 Testing Smoothness

In this section we describe how one can test in parallel polynomial time whether a closed projective variety X is smooth.

Crucial is Proposition 3.8 implying that if X is smooth, then X is scheme-theoretically cut out by hypersurfaces of degree $\leq D = \deg X$. Using the linear system of equations (15) one can compute a vector space basis $f_1, \ldots, f_N$ of $I_{\leq D}$, where $I = I(X)$. Let U_i be the open subset $X \setminus \mathcal{Z}(X_i)$ for $0 \leq i \leq n$. Then (cf. §2.1) the tangent space of X at each $x \in U_i$ is

$$T_xX = \mathcal{Z}(d_xf_1^i, \ldots, d_xf_N^i) \subseteq \mathbb{C}^n \simeq \mathbb{P}^n \setminus \mathcal{Z}(X_i).$$

Hence, if we assume X to be m-equidimensional and denote with L_x^i the linear subspace $\mathcal{Z}(d_xf_1^i, \ldots, d_xf_N^i)$, we have

$$X \text{ smooth} \quad \Longleftrightarrow \quad \bigwedge_i \forall\, x \in U_i \;\; \dim L_x^i = m, \tag{18}$$

which is the Jacobian criterion. Indeed, if X is not smooth at $x \in U_i$, then $\dim L_x^i \geq \dim T_xX > m$, since in general $T_xX \subseteq L_x^i$.

Now our algorithm reads as follows.

Algorithm Smoothness Test

`input` X given by homogeneous polynomials of degree $\leq d$.

1. Compute the equidimensional decomposition $X = Z_0 \cup \cdots \cup Z_n$, where Z_m is either empty or m-equidimensional.

2. `if` $Z_n \neq \emptyset$ `then output` "Yes".

3. `for` $0 \leq m < m' < n$ `do if` $Z_m \cap Z_{m'} \neq \emptyset$ `then output` "No".

4. Set $D := d^n$.

5. Compute a basis $f_1, \ldots, f_N$ of $I_{\leq D}$, where $I = I(X)$.

6. `for` $0 \le m < n$ `do`

7. `for` $0 \le i \le n$ `do`

8. Compute the matrix $A := \left(\frac{\partial f_\nu^i}{\partial X_\mu}\right)_{\mu,\nu}$, where f_ν^i is the dehomogenization of f_ν with respect to X_i.

9. Compute the Mulmuley polynomial $p(Z)$ of A (see §2.7), which lies in $\mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n, T, Z]$. Write $p(Z) = p_0 + p_1 Z + \cdots + p_K Z^K$, and let $F_1, \ldots, F_L \in \mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]$ be the coefficients of all T^k in $p_0, \ldots, p_m$.

10. `if` $Z_m \cap \mathcal{Z}(F_1, \ldots, F_L) \cap \{X_i \neq 0\} \neq \emptyset$ `then output` "No".

11. `output` "Yes".

Proposition 6.1. *The algorithm Smoothness Test is correct and can be implemented in parallel time* $(n \log d)^{\mathcal{O}(1)}$ *and sequential time* $d^{\mathcal{O}(n^4)}$.

Proof. Correctness: If X is smooth, then it clearly passes the test in step 3. By (18) we have $\dim L_x^i = m$ for all m, i and all $x \in Z_m$. Denote by A_x the matrix A evaluated at x, and similarly for $p(Z)$. Then $L_x^i = \ker A_x$, hence

$$\dim L_x^i > m \quad \Longleftrightarrow \quad Z^{m+1} \,|\, p_x(Z) \quad \Longleftrightarrow \quad F_1(x) = \cdots = F_L(x) = 0.$$

If X is not smooth, then it doesn't pass the test in step 3 or some Z_m is not smooth. In the latter case at some point $x \in Z_m \cap U_i$ we will have $\dim L_x^i > m$.

Analysis: All the algorithms we use are well-parallelizable. We therefore state only the sequential time bounds. The equidimensional decomposition in step 1 can be done in time $d^{\mathcal{O}(n^2)}$ with the algorithm of [27]. For each m, this algorithm returns $d^{\mathcal{O}(n^2)}$ polynomials of degree bounded by $\deg Z_m = \mathcal{O}(d^n)$ whose zero set is Z_m. Testing feasibility of a system of r homogeneous equations of degree $\overline{d}$ can be done in time $r(n\overline{d})^{\mathcal{O}(n)}$ using the effective homogeneous Nullstellensatz. Hence step 3 takes time $d^{\mathcal{O}(n^2)}$. Szántó's algorithm in step 5 runs in time $d^{\mathcal{O}(n^4)}$, and clearly $N = \mathcal{O}(D^{n+1}) = d^{\mathcal{O}(n^2)}$. Furthermore, the computation of the Mulmuley polynomial in step 9 can be done in time $d^{\mathcal{O}(n^3)}$ by §2.7, and we have $L = \mathcal{O}(N^2 m) = d^{\mathcal{O}(n^2)}$ and $\deg F_i \le ND = d^{\mathcal{O}(n^2)}$. Thus step 10 takes time $d^{\mathcal{O}(n^4)}$ by the affine effective Nullstellensatz. □

7 Finding Generic Hyperplanes

The algorithm for computing the cohomology of X described in §4.2 and §5 depends on a choice of sufficiently generic hyperplane sections H_ν of X. More precisely, it is required that $V := H_0 \cup \cdots \cup H_m$ is a hypersurface with normal crossings in X, where $m = \dim X$. Note that as a consequence $H_{i_0} \cup \cdots \cup H_{i_q}$ has normal crossings for each tuple $i_0 < \cdots < i_q$. Here we describe how to find sufficiently generic hyperplanes deterministically in parallel polynomial time.

Throughout this section we assume X to be smooth, and let us first assume that X is m-equidimensional. We will formulate a sufficient condition for normal crossings in terms of transversality. Recall that a linear subspace $L \subseteq \mathbb{P}^n$ is called *transversal* to X in $x \in X \cap L$, written $X \pitchfork_x L$, iff $\dim(T_xX \cap T_xL) = \dim T_xX + \dim T_xL - n$. Now let the hypersurfaces $H_0, \ldots, H_m$ be given by the linear forms $\ell_0, \ldots, \ell_m \in \mathbb{C}[X_0, \ldots, X_n]$. Denote $L_{i_0 \cdots i_q} := \mathcal{Z}(\ell_{i_0}, \ldots, \ell_{i_q})$ for all $0 \leq q \leq m$ and all $0 \leq i_0 < \cdots < i_q \leq m$.

Lemma 7.1. *If $\ell_0, \ldots, \ell_m$ are linearly independent and for all $0 \leq q \leq m$ and all $0 \leq i_0 < \cdots < i_q \leq m$ we have*

$$\forall x \in X \cap L_{i_0 \cdots i_q} \colon \; X \pitchfork_x L_{i_0 \cdots i_q}, \tag{19}$$

then $V \subseteq X$ is a hypersurface with normal crossings.

Proof. Suppose that the condition (19) holds. First note by choosing $q = 0$ that $L_i = \mathcal{Z}(\ell_i)$ is transversal to X at all x, thus V is indeed a hypersurface. Furthermore, $H_i = X \cap L_i$ is smooth in x, so that x lies in only one irreducible component of H_i, and $\ell_i \in \mathcal{O}_{X,x}$ is a local equation of that component. By transversality we have $\dim(T_xX \cap T_xL_{i_0 \cdots i_q}) = m - q - 1$. But $T_xX \cap T_xL_{i_0 \cdots i_q}$ is the kernel of the linear map $\varphi := (d_x\ell_{i_0}, \ldots, d_x\ell_{i_q}) \colon T_xX \to \mathbb{C}^{q+1}$, which thus must be surjective. Hence $d_x\ell_{i_0}, \ldots, d_x\ell_{i_q}$ are linearly independent on T_xX, which proves the claim. □

In order to work with condition (19) algorithmically, we introduce some notation. Set $I := I(X)$ and $D := \deg X$. Recall from §6 that if $f_1, \ldots, f_N$ is a vector space basis of $I_{\leq D}$, then

$$T_xX = \mathcal{Z}(d_xf_1^i, \ldots, d_xf_N^i) \subseteq \mathbb{C}^n$$

for all $x \in U_i = X \setminus \mathcal{Z}(X_i)$ and $0 \leq i \leq n$. For each tuple $i_0 < \cdots < i_q$ and each i we define the matrix

$$A^i_{i_0 \cdots i_q} := \begin{pmatrix} d_xf_1^i \\ \vdots \\ d_xf_N^i \\ d_x\ell_{i_0}^i \\ \vdots \\ d_x\ell_{i_q}^i \end{pmatrix} \in \mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]^{(N+q+1) \times n}. \tag{20}$$

Then the kernel of $A^i_{i_0 \cdots i_q}$ is the kernel of φ of the proof of Lemma 7.1. Assume that $\ell_0, \ldots, \ell_m$ are linearly independent. Then condition (19) is eqivalent to the statement that the nullity of $A^i_{i_0 \cdots i_q}$ is $m - q - 1$, its minimal possible value, at each point $x \in U_i \cap L_{i_0 \cdots i_q}$. Note that this condition also implies the linear independence. Now let $p(Z)$ be the Mulmuley polynomial of $A^i_{i_0 \cdots i_q}$, which lies in $\mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n, T, Z]$. Let $F_1, \ldots, F_L \in \mathbb{C}[X_0, \ldots, \widehat{X_i}, \ldots, X_n]$ be the coefficients of all T^k in the coefficient of Z^{m-q} in $p(Z)$. Then a sufficient condition for (19) is

$$\bigwedge_i U_i \cap L_{i_0 \cdots i_q} \cap \mathcal{Z}(F_1, \ldots, F_L) \neq \emptyset. \tag{21}$$

Using this formula we can prove

Proposition 7.2. *Given polynomials of degree $\leq d$ defining a smooth subvariety $X \subseteq \mathbb{P}^n$ of dimension m, one can compute in parallel time $(n \log d)^{\mathcal{O}(1)}$ and sequential time $d^{\mathcal{O}(n^4)}$ linear forms $\ell_0, \ldots, \ell_m$ such that $V = \bigcup_j H_j$ is a hypersurface with normal crossings, where $H_j = \mathcal{Z}_X(\ell_j)$.*

Proof. First we set $D := d^n$ and compute a basis $f_1, \ldots, f_N$ of I_D. Then we compute the equidimensional components Z_m of X. We find the linear forms $\ell_0, \ldots, \ell_m$ successively, one at a time. So assume that $\ell_0, \ldots, \ell_{j-1}$ have been already found, and take $\ell_j = \alpha_0 X_0 + \cdots + \alpha_n X_n$ with indeterminate coefficients $\alpha = (\alpha_0, \ldots, \alpha_n)$. Now consider the conjunction of the conditions (21) for all $m \leq \dim X$ and $i_0 < \cdots < i_q = j$, which is a first order formula with free variables α. Note that here one has to take $U_i = Z_m \cap \{X_i \neq 0\}$. By quantifier elimination compute an equivalent quantifier-free formula $\Phi(\alpha)$ in disjunctive normal form. Let $G_1, \ldots, G_M$ be all polynomials accuring $\Phi(\alpha)$. Since $\Phi(\alpha)$ is satisfied for generic α, it is easy to see that $G_\nu(\alpha) \neq 0$ for all ν implies $\Phi(\alpha)$ [13, proof of Theorem 3.8]. Let δ be the maximal degree of the G_ν. Now take a set $S \subseteq \mathbb{C}$ of cardinality $> Mn\delta$ and test for all $b \in S$ in parallel, whether $P_\nu(b) := G_\nu(1, b, \ldots, b^n) \neq 0$ for all $1 \leq \nu \leq M$. Since P_ν is a univariate polynomial of degree $\leq n\delta$, there must exist a successful b. Then we can take $\ell_j = X_0 + bX_1 + \ldots + b^n X_n$.

Analysis: The computation of $f_1, \ldots, f_N$, of the equidimensional decomposition, and of the Mulmuley polynomial can be done within the claimed time bounds. Recall that N, L, the degrees of the defining equations for Z_m, as well as $\deg F_i$ are of order $d^{\mathcal{O}(n^2)}$. Condition (21) is a universal first order formula with $\mathcal{O}(n)$ free and bounded variables, and $d^{\mathcal{O}(n^2)}$ atomic formulas involving polynomials of degree $d^{\mathcal{O}(n^2)}$. According to [23], one can eliminate the universal quantifier and hence compute the polynomials G_ν in parallel time $(n \log d)^{\mathcal{O}(1)}$ and sequential time $d^{\mathcal{O}(n^4)}$. Furthermore, M and δ are also bounded by $d^{\mathcal{O}(n^4)}$. Hence the cardinality of the set S is $d^{\mathcal{O}(n^4)}$ and our claim follows. □

Theorem 1.1 follows from the Propositions 6.1, 7.2, and 5.3.

References

[1] D. Arapura. Frobenius amplitude and strong vanishing theorems for vector bundles. *Duke Math. J.*, 121(2):231–267, 2004. With an appendix by Dennis S. Keeler.

[2] M.F. Atiyah, R. Bott, and L. Gårding. Lacunas for hyperbolic differential operators with constant coefficients. II. *Acta Math.*, 131:145–206, 1973.

[3] S. Basu. Computing the first few Betti numbers of semi-algebraic sets in single exponential time. *J. Symbolic Comput.*, 41(10):1125–1154, 2006.

[4] S. Basu. Algorithmic semi-algebraic geometry and topology – recent progress and open problems. In *Surveys on discrete and computational geometry*, volume 453 of *Contemp. Math.*, pages 139–212. Amer. Math. Soc., Providence, RI, 2008.

[5] D. Bayer and D. Mumford. What can be computed in algebraic geometry? In *Computational algebraic geometry and commutative algebra (Cortona, 1991)*, Sympos. Math., XXXIV, pages 1–48. Cambridge Univ. Press, Cambridge, 1993.

[6] D. Bayer and M. Stillman. A criterion for detecting m-regularity. *Invent. Math.*, 87:1–11, 1987.

[7] S.J. Berkowitz. On computing the determinant in small parallel time using a small number of processors. *Inf. Process. Lett.*, 18(3):147–150, 1984.

[8] A. Bertram, L. Ein, and R. Lazarsfeld. Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *J. Amer. Math. Soc.*, 4(3):587–602, 1991.

[9] N. Bourbaki. *Elements of mathematics. Algebra, Part I: Chapters 1-3.* Hermann, Paris, 1974. Translated from the French.

[10] W.D. Brownawell. Bounds for the degrees in the Nullstellensatz. *Ann. of Math. (2)*, 126(3):577–591, 1987.

[11] P. Bürgisser and F. Cucker. Variations by complexity theorists on three themes of Euler, Bézout, Betti, and Poincaré. In J. Krajíček, editor, *Complexity of computations and proofs*, volume 13 of *Quaderni di Matematica [Mathematics Series]*, pages 73–152. Department of Mathematics, Seconda Università di Napoli, Caserta, 2004.

[12] P. Bürgisser and P. Scheiblechner. On the complexity of counting components of algebraic varieties. *J. Symb. Comp.*, 44(9):1114–1136, 2009.

[13] P. Bürgisser and P. Scheiblechner. Counting irreducible components of complex algebraic varieties. *Comp. Compl.*, 19(1):1–35, 2010.

[14] L. Caniglia, A. Galligo, and J. Heintz. Equations for the projective closure and effective Nullstellensatz. *Discr. Appl. Math.*, 33(1-3):11–23, 1991.

[15] P. Deligne. *Équations différentielles à points singuliers réguliers.* Lecture Notes in Mathematics, Vol. 163. Springer-Verlag, Berlin, 1970.

[16] P. Deligne and A. Dimca. Filtrations de Hodge et par l'ordre du pôle pour les hypersurfaces singulières. *Annales scientifiques de l'École Normale Supérieure*, 23(4):645–656, 1990.

[17] A. Dimca. *Singularities and Topology of Hypersurfaces.* Universitext. Springer Verlag, 1992.

[18] D. Eisenbud. *Commutative Algebra with a View Toward Algebraic Geometry*, volume 150 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[19] D. Eisenbud, G. Fløystad, and F.-O. Schreyer. Sheaf cohomology and free resolutions over exterior algebras. *Trans. Amer. Math. Soc.*, 355(11):4397–4426 (electronic), 2003.

[20] D. Eisenbud and S. Goto. Linear free resolutions and minimal multiplicity. *J. Algebra*, 88(1):89–133, 1984.

[21] D. Eisenbud, D. Grayson, M. Stillman, and B. Sturmfels (Eds.). *Computations in algebraic geometry with Macaulay 2.* Number 8 in Algorithms and Computations in Mathematics. Springer Verlag, 2001.

[22] N. Fitchas and A. Galligo. Nullstellensatz effectif et conjecture de Serre (théorème de Quillen-Suslin) pour le calcul formel. *Math. Nachr.*, 149:231–253, 1990.

[23] N. Fitchas, A. Galligo, and J. Morgenstern. Precise sequential and parallel complexity bounds for quantifier elimination over algebraically closed fields. *J. Pure Appl. Alg.*, 67:1–14, 1990.

[24] A. Galligo. Théorème de division et stabilité en géométrie analytique locale. *Ann. Inst. Fourier (Grenoble)*, 29(2):vii, 107–184, 1979.

[25] D. Giaimo. On the Castelnuovo-Mumford regularity of connected curves. *Trans. Amer. Math. Soc.*, 358(1):267–284 (electronic), 2006.

[26] M. Giusti. Some effectivity problems in polynomial ideal theory. In *EUROSAM '84: Proceedings of the International Symposium on Symbolic and Algebraic Computation*, pages 159–171, London, UK, 1984. Springer-Verlag.

[27] M. Giusti and J. Heintz. Algorithmes -disons rapides- pour la décomposition d'une variété algébrique en composantes irréductibles et équidimensionnelles. In T. Mora C. Traverso, editor, *Effective Methods in Algebraic Geometry (Proceedings of MEGA'90)*, volume 94 of *Progress in Math.*, pages 169–193, New York, NY, USA, 1991. Birkhäuser.

[28] D.R. Grayson and M.E. Stillman. Macaulay2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2/.

[29] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley, New York, 1978.

[30] A. Grothendieck. On the de Rham cohomology of algebraic varieties. *Publications Mathématiques IHES*, 39:93–103, 1966.

[31] L. Gruson, R. Lazarsfeld, and C. Peskine. On a theorem of Castelnuovo, and the equations defining space curves. *Invent. Math.*, 72(3):491–506, 1983.

[32] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977.

[33] Z. Jelonek. On the effective Nullstellensatz. *Invent. Math.*, 162(1):1–17, 2005.

[34] J. Kollár. Sharp effective Nullstellensatz. *J. Amer. Math. Soc.*, 1(4):963–975, 1988.

[35] S. Kwak. Generic projections, the equations defining projective varieties and Castelnuovo regularity. *Math. Z.*, 234(3):413–434, 2000.

[36] D. Lazard. Resolution des systemes d'equations algebriques. *Theoret. Comp. Sci.*, 15:77–110, 1981.

[37] R. Lazarsfeld. A sharp Castelnuovo bound for smooth surfaces. *Duke Math. J.*, 55(2):423–429, 1987.

[38] R. Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[39] E.W. Mayr. Some complexity results for polynomial ideals. *J. Compl.*, 13(3):303–325, 1997.

[40] E.W. Mayr and A.R. Meyer. The complexity of the word problems for commutative semigroups and polynomial ideals. *Advances in Mathematics*, 46(3):305–329, 1982.

[41] J. McCleary. *A User's Guide to Spectral Sequences*, volume 12 of *Mathematics Lecture Series.* Publish or Perish, Inc, Wilmington, Delaware, 1985.

[42] K. Mulmuley. A fast parallel algorithm to compute the rank of a matrix over an arbitrary field. *Combinatorica*, 7(1):101–104, 1987.

[43] D. Mumford. *Lectures on Curves on an Algebraic Surface.* With a section by G. M. Bergman, Annals of Mathematics Studies, No. 59. Princeton University Press, Princeton, N.J., 1966.

[44] D. Mumford. Varieties defined by quadratic equations. In *Questions on Algebraic Varieties (C.I.M.E., III Ciclo, Varenna, 1969)*, pages 29–100. Edizioni Cremonese, Rome, 1970.

[45] D. Mumford. *Algebraic Geometry I: Complex Projective Varieties*, volume 221 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, Berlin Heidelberg New York, 1976.

[46] T. Oaku and N. Takayama. An algorithm for de Rham cohomology groups of the complement of an affine variety via D-module computation. *Journal of Pure and Applied Algebra*, 139:201–233, 1999.

[47] I. Peeva and B. Sturmfels. Syzygies of codimension 2 lattice ideals. *Math. Z.*, 229(1):163–194, 1998.

[48] H.C. Pinkham. A Castelnuovo bound for smooth surfaces. *Invent. Math.*, 83(2):321–332, 1986.

[49] Z. Ran. Local differential geometry and generic projections of threefolds. *J. Differential Geom.*, 32(1):131–137, 1990.

[50] P. Scheiblechner. On the complexity of counting irreducible components and computing Betti numbers of complex algebraic varieties. PhD Thesis, 2007.

[51] P. Scheiblechner. On the complexity of deciding connectedness and computing Betti numbers of a complex algebraic variety. *J. Compl.*, 23(3):359–379, 2007.

[52] P. Scheiblechner. On a generalization of Stickelberger's theorem. *J. Symb. Comp.*, 45(12):1459 – 1470, 2010. MEGA'2009.

[53] J.P. Serre. Géométrie algébrique et géométrie analytique. *Ann. Inst. Fourier, Grenoble*, 6:1–42, 1955–1956.

[54] G. Smith. Computing global extension modules,. *J. Symb. Comp.*, 29:729–746, 2000.

[55] J. Stückrad and W. Vogel. Castelnuovo's regularity and multiplicity. *Math. Ann.*, 281(3):355–368, 1998.

[56] Á. Szántó. Complexity of the Wu-Ritt decomposition. In *PASCO '97: Proceedings of the second international symposium on Parallel symbolic computation*, pages 139–149, New York, NY, USA, 1997. ACM Press.

[57] Á. Szántó. Computation with polynomial systems. PhD Thesis, 1999.

[58] W.V. Vasconcelos. *Computational methods in commutative algebra and algebraic geometry*, volume 2 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 1998.

[59] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

[60] J. von zur Gathen. Parallel arithmetic computations: a survey. In *MFOCS86*, number 233 in LNCS, pages 93–112. SV, 1986.

[61] U. Walther. Algorithmic computation of de Rham cohomology of complements of complex affine varieties. *J. Symb. Comp.*, 29(4-5):795–839, 2000.

[62] U. Walther. Algorithmic determination of the rational cohomology of complex varieties via differential forms. In *Symbolic computation: solving equations in algebra, geometry, and engineering (South Hadley, MA, 2000)*, volume 286 of *Contemp. Math.*, pages 185–206. Amer. Math. Soc., Providence, RI, 2001.