

Effective approximation of the solutions of algebraic equations

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Abstract. Let F be a holomorphic map whose components satisfy some polynomial relations. We present an algorithm for constructing Nash maps locally approximating F , whose components satisfy the same relations.

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1 Introduction

Let Q_1, \dots, Q_q be polynomials in \hat{m} complex variables and let $F : U \rightarrow \mathbf{C}^{\hat{m}}$ be a holomorphic map such that

$$Q_1(F(x)) = \dots = Q_q(F(x)) = 0$$

for every $x \in U$, where U is a domain in \mathbf{C}^n . Let us fix any point $x_0 \in U$. The aim of this paper is to present an algorithm for constructing a Nash map F^ν uniformly approximating F , in some neighborhood U_0 of x_0 , such that

$$Q_1(F^\nu(x)) = \dots = Q_q(F^\nu(x)) = 0$$

for every $x \in U_0$. The correctness of the algorithm will follow from the proof of the fact that such approximations always exist, presented in Section 3.1 (see Theorem 3.1).

The existence of local approximation of the solutions of algebraic or analytic equations was investigated in [1], [2] and [3], and Theorem 3.1 can be derived from the results of these papers. Also, a more general global version of Theorem 3.1 is known to be true (see [24], pp 338-339, for a proof based on general Néron desingularization, and therefore not suitable for effective constructions). In order to obtain an effective procedure for constructing the approximating functions it is more convenient to base a proof on a combination of some of the ideas of [1], [34] and [5]. This allows us to perform an effective reduction to the case where $q = 1$ and Q_1 is of the form

$$Q_1(y_1, \dots, y_{\hat{m}}) = y_{\hat{m}}^s + y_{\hat{m}}^{s-1} p_1(y_1, \dots, y_{\hat{m}-1}) + \dots + p_s(y_1, \dots, y_{\hat{m}-1}),$$

where $s \in \mathbf{N}$ and $p_1, \dots, p_s \in \mathbf{C}[y_1, \dots, y_{\hat{m}-1}]$ and, moreover, $\frac{\partial Q_1}{\partial y_{\hat{m}}}(F(x))$ is a non-zero function. Then we can use the fact that in order to find Nash approximations $F^\nu : U_0 \rightarrow \mathbf{C}^{\hat{m}}$ of $F|_{U_0}$ such that $Q_1(F^\nu(x)) = 0$ it is sufficient to find

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Nash approximations \bar{F}^ν of $F|_{U_0}$ such that $Q_1(\bar{F}^\nu(x)) = d(x)(\frac{\partial Q_1}{\partial y_{\hat{m}}}(\bar{F}^\nu(x)))^2$, where $d(x)$ is a holomorphic function with sufficiently small norm. This follows from the Taylor expansion of Q_1 about $\bar{F}^\nu(x)$ with respect to $y_{\hat{m}}$. More precisely, introduce a new variable Y and compute

$$\begin{aligned} & Q_1(\bar{F}^\nu(x) + (0, \dots, 0, d(x)\frac{\partial Q_1}{\partial y_{\hat{m}}}(\bar{F}^\nu(x))Y)) = \\ & Q_1(\bar{F}^\nu(x)) + d(x)(\frac{\partial Q_1}{\partial y_{\hat{m}}}(\bar{F}^\nu(x)))^2 Y + d(x)(\frac{\partial Q_1}{\partial y_{\hat{m}}}(\bar{F}^\nu(x)))^2 \cdot \text{higher order terms} = \\ & = d(x)(\frac{\partial Q_1}{\partial y_{\hat{m}}}(\bar{F}^\nu(x)))^2 \cdot (1 + Y + \text{higher order terms}). \end{aligned}$$

The "higher order terms" can be estimated so that there is a holomorphic function $Y = Y(x)$ for which the expression in the last bracket vanishes. Then $\bar{F}^\nu(x) + (0, \dots, 0, d(x)\frac{\partial Q_1}{\partial y_{\hat{m}}}(\bar{F}^\nu(x))Y(x)$ is a solution we look for. This fact was proved in detail in [34] and applied in a similar context to (global) maps depending on one variable (cf. [34]). If the number n (of the variables F depends on) is greater than 1, then we decrease it using the Weierstrass Division Theorem in the spirit of [1], see also [13] pp. 295-298 or [29] pp. 97-98.

To perform an algorithm we need to represent holomorphic functions in such a way that they can constitute its input. This will be done similarly to [36], [38]. More precisely, every component f of F will be defined by the following data:

- (a) a procedure computing the coefficients of the Taylor expansion of f at x_0 ,
- (b) the polyradius of a polydisc U_f centered at x_0 on which the Taylor expansion of f is convergent,
- (c) a constant M_f such that $\sup_{U_f} |f| < M_f$.

The class of functions represented in this way is large and it contains important transcendental functions (such as \exp , \log , \sin , \cos and many others). Moreover, it is closed under algebraic operations and differentiation which can be performed effectively. Furthermore, one can use computer algebra procedures [15] to obtain effective versions of the Weierstrass Preparation and Division Theorems. Using basic facts from complex analysis (Cauchy's integral formula), it is also possible to control accuracy of polynomial approximations of such functions (and, consequently, accuracy of Nash approximations of F ; see Section 3.2 below for details).

In Section 3.2 we present algorithms for computing approximating maps. The algorithms rely on effective versions of the Weierstrass Preparation and Division Theorems and on some techniques from computational algebraic geometry (equidimensional decomposition [15], [18], [19], [23], elimination theory [12], [14], [15], description of factors of iterated discriminants [22]).

This paper is related to [5] where also a method for approximation of holomorphic solutions of polynomial equations is presented. However, the method of [5] contains steps which are not computable in the data structure described

above. More precisely, it relies on the zero-test and on factorization of monic polynomials with holomorphic coefficients into the product of powers of reduced monic and pairwise relatively prime polynomials with holomorphic coefficients. (The zero-test is only semi-decidable, and computability of the factorization would imply decidability of the zero-test, cf. Remark 3.8 below.) The approach proposed in the present paper allows us to avoid these steps. First, in Section 3.2, we give a partial algorithm which does not rely on the factorization of polynomials with holomorphic coefficients, but still uses the zero-test. Next after refining this partial algorithm, we obtain a complete one.

Our work is motivated by the natural question whether every purely dimensional analytic set can be (locally) approximated by algebraic or Nash ones of the same pure dimension. This question is related to the classical problem of characterizing those analytic sets which are analytically equivalent to algebraic ones (see e.g. [1], [2], [8], [9], [24]). As for algebraic approximation, the problem is interesting especially for non-complete intersections, i.e. for analytic sets for which the number of defining equations is greater than their codimensions. Then one cannot simply replace the defining functions by approximating polynomials as this gives sets of strictly smaller dimensions. Nevertheless, algebraic approximations do exist for a large subclass of the class of non-complete intersections (see [5], [6], [7]). In particular, every analytic set admits local algebraic approximations which in many cases can be effectively constructed. One of the main tools used in the construction is Theorem 3.1 discussed in the present paper (cf. [6], p. 284) and the algorithm following from its proof.

Finally, let us mention that both in the real case [11] and in the complex case [24] global versions of Theorem 3.1 are known to be true. The original proofs of these theorems are elegant and relatively short, but rely on the affirmative solution to Artin's conjecture—a deep and difficult result of commutative algebra (cf. [11], [24]). However, these proofs do not indicate any simple constructive procedures of approximation.

The present paper is organized as follows. Sections 3.1 and 3.2 are devoted to Theorem 3.1 and the algorithms, respectively. Preliminary material concerning Nash mappings and sets as well as analytic sets with proper projection is gathered in Section 2 below.

2 Preliminaries

2.1 Nash mappings and sets

Let Ω be an open subset of \mathbf{C}^n and let f be a holomorphic function on Ω . We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \rightarrow \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω . A holomorphic mapping defined on Ω with values in \mathbf{C}^N is said to be a Nash mapping if each of its components is a Nash function.

The following lemma is well known. Here we recall the proof to emphasize its algorithmic nature.

Lemma 2.1 *Let $f, g : \Omega \rightarrow \mathbf{C}$ be Nash functions and let $P \in \mathbf{C}[x, y], Q \in \mathbf{C}[x, z]$ be monic in y, z , respectively, such that*

$$P(x, f(x)) = 0, Q(x, g(x)) = 0 \text{ for } x \in \Omega.$$

Then there are $S, R \in \mathbf{C}[x, u]$, monic in u , such that

$$S(x, (f + g)(x)) = 0, R(x, (f \cdot g)(x)) = 0 \text{ for } x \in \Omega.$$

Given P, Q , one can effectively compute S, R . Moreover, the degrees of S, R in u are bounded by a constant depending only on the degrees of P, Q in y, z .

Proof. Let \diamond denote either multiplication or addition in \mathbf{C} . Let $\pi : \mathbf{C}_x^n \times \mathbf{C}_u \times \mathbf{C}_y \times \mathbf{C}_z \rightarrow \mathbf{C}_x^n \times \mathbf{C}_u$ denote the natural projection. Consider the algebraic set

$$V = \{(x, u, y, z) \in \mathbf{C}^n \times \mathbf{C} \times \mathbf{C} \times \mathbf{C} : P(x, y) = Q(x, z) = 0, u = y \diamond z\}.$$

Since P, Q are monic in y, z , and $u - y \diamond z$ is monic in u we see that the map $\pi|_V : V \rightarrow \mathbf{C}_x^n \times \mathbf{C}_u$ is proper and also the projection of $\pi(V)$ onto \mathbf{C}_x^n is proper. Consequently $\pi(V)$ is an algebraic hypersurface described by a single polynomial denoted by $S(x, u)$ (if $\diamond = +$) or $R(x, u)$ (if $\diamond = \cdot$). Clearly, $S(x, (f + g)(x)) = 0$ and $R(x, (f \cdot g)(x)) = 0$. Observe that, by properness of the projection of $\pi(V)$ onto \mathbf{C}_x^n , the polynomials S, R are monic in u .

Note that S, R can be effectively computed and the degrees in u of S, R are bounded by a constant depending only on the degrees of P, Q in y, z , respectively. This follows by the fact that $S(x, u), R(x, u)$ are obtained by taking the resultants of pairs of polynomials in y with coefficients in $\mathbf{C}[x, u]$. Namely, $S(x, u) = \text{res}_y(P(x, y), Q(x, u - y))$ and $R(x, u) = \text{res}_y(P(x, y), y^d Q(x, \frac{u}{y}))$, where d is the degree of Q in z .

In other words, the projection of V to $\mathbf{C}_x^n \times \mathbf{C}_u$ corresponds to the elimination of y, z from $P, Q, u - y \diamond z$, which is algorithmic (see also [4], [12], Chapter 3, or [15], pp. 69-73) .■

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions f_1, \dots, f_s on U such that

$$Y \cap U = \{x \in U : f_1(x) = \dots = f_s(x) = 0\}.$$

We will use the following fact from [32], p. 239. Let $\pi : \Omega \times \mathbf{C}^k \rightarrow \Omega$ denote the natural projection.

Theorem 2.2 *Let X be a Nash subset of $\Omega \times \mathbf{C}^k$ such that $\pi|_X : X \rightarrow \Omega$ is a proper mapping. Then $\pi(X)$ is a Nash subset of Ω and $\dim(X) = \dim(\pi(X))$.*

The fact from [32] stated below explains the relation between Nash and algebraic sets.

Theorem 2.3 *Let X be a Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then every analytic irreducible component of X is an irreducible Nash subset of Ω . Moreover, if X is irreducible then there exists an algebraic subset Y of \mathbf{C}^n such that X is an analytic irreducible component of $Y \cap \Omega$.*

Corollary 2.4 *Let U be an open subset of \mathbf{C}^n . Let $f : U \rightarrow \mathbf{C}$ be a holomorphic function and let $N \subset U \times \mathbf{C}$ be a Nash hypersurface such that $\text{graph}(f) \subset N$. Then f is a Nash function.*

Proof. One may assume that U is connected (because each of its connected components can be considered separately). If N is reducible, then replace it by its analytic irreducible component containing $\text{graph}(f)$ which, by Theorem 2.3, is a Nash set. Again by Theorem 2.3, there is an algebraic hypersurface $M \subset \mathbf{C}^n \times \mathbf{C}$ such that $N \subset M$. Let $P \in \mathbf{C}[x_1, \dots, x_n, z]$ be a (non-zero) polynomial such that $\{P = 0\} = M$. Then $P(x, f(x)) = 0$ for every $x \in U$. ■

2.2 Analytic sets

For any map f , by $\text{graph}(f)$ we denote the set of all (x, y) such that $f(x) = y$.

Let U, U' be domains in $\mathbf{C}^n, \mathbf{C}^k$, respectively, and let $\pi : \mathbf{C}^n \times \mathbf{C}^k \rightarrow \mathbf{C}^n$ denote the natural projection. For any purely n -dimensional analytic subset Y of $U \times U'$ such that $\pi|_Y : Y \rightarrow U$ is a proper mapping, we denote by $\mathcal{S}(Y, \pi)$ the set of singular points of $\pi|_Y$:

$$\mathcal{S}(Y, \pi) = \text{Sing}(Y) \cup \{x \in \text{Reg}(Y) : (\pi|_Y)'(x) \text{ is not an isomorphism}\}.$$

We often write $\mathcal{S}(Y)$ instead of $\mathcal{S}(Y, \pi)$ when it is clear which projection is taken into consideration.

It is well known that $\mathcal{S}(Y)$ is an analytic subset of $U \times U'$, $\dim(\mathcal{S}(Y)) < n$ (cf. [10], p. 50), hence, by the Remmert Theorem $\pi(\mathcal{S}(Y))$ is also analytic. Moreover, the following holds. The mapping $\pi|_Y$ is surjective and open and there exists an integer $s = s(\pi|_Y)$ such that:

- (1) $\#(\pi|_Y)^{-1}(\{a\}) < s$ for $a \in \pi(\mathcal{S}(Y))$,
- (2) $\#(\pi|_Y)^{-1}(\{a\}) = s$ for $a \in U \setminus \pi(\mathcal{S}(Y))$,
- (3) for every $a \in U \setminus \pi(\mathcal{S}(Y))$ there exist a neighborhood W of a and holomorphic mappings $f_1, \dots, f_s : W \rightarrow U'$ such that $\text{graph}(f_i) \cap \text{graph}(f_j) = \emptyset$ for $i \neq j$ and $\text{graph}(f_1) \cup \dots \cup \text{graph}(f_s) = (W \times U') \cap Y$.

Let E be a purely n -dimensional analytic subset of $U \times U'$ with proper projection onto a domain $U \subset \mathbf{C}^n$, where U' is a domain in \mathbf{C} . Then there is a monic polynomial $p \in \mathcal{O}(U)[z]$ (i.e. a polynomial in z whose leading coefficient is 1) such that $E = \{(x, z) \in U \times \mathbf{C} : p(x, z) = 0\}$ and the discriminant Δ_p of p is not identically zero. p will be called the optimal polynomial for E . We have $\tilde{\pi}(\mathcal{S}(E)) = \{x \in U : \Delta_p(x) = 0\}$, where $\tilde{\pi} : U \times \mathbf{C} \rightarrow U$ is the natural projection. If E is algebraic and $U = \mathbf{C}^n, U' = \mathbf{C}$, then the coefficients of the optimal polynomial p are polynomials.

Let V be an analytic subset of $\mathbf{C}^n \times \mathbf{C}^k$ with proper projection onto \mathbf{C}^n , let $\Phi_L : \mathbf{C}^n \times \mathbf{C}^k \rightarrow \mathbf{C}^n \times \mathbf{C}$ be given by $\Phi_L(u, v) = (u, L(v))$, where L is \mathbf{C} -linear. Then $\Phi_L|_V : V \rightarrow \mathbf{C}^n \times \mathbf{C}$ is a proper map. Indeed, fix any compact subset K of $\mathbf{C}^n \times \mathbf{C}$. Let K' be the image of the projection of K onto \mathbf{C}^n . Then $(\Phi_L|_V)^{-1}(K)$ is clearly closed (in the Euclidean topology). It is also bounded because $(\Phi_L|_V)^{-1}(K) \subset (K' \times \mathbf{C}^k) \cap V$, so $\Phi_L|_V$ is proper. Consequently, $\Phi_L(V)$ is analytic (or algebraic if V is such). Also observe that if generic fibers in $\Phi_L(V)$ and in V over \mathbf{C}^n have the same number of elements, then $\Phi_L(\mathcal{S}(V)) \subset \mathcal{S}(\Phi_L(V))$. Indeed, pick any $(a, b) \in \mathcal{S}(V) \subset \mathbf{C}^n \times \mathbf{C}^k$ and a polydisc $P_1 \times P_2 \subset \mathbf{C}^n \times \mathbf{C}^k$ such that $V \cap (P_1 \times P_2)$ has proper projection onto P_1 and $V \cap (\{a\} \times P_2)$ is a one element set. Then, by definition of $\mathcal{S}(V)$, generic fibers in $V \cap (P_1 \times P_2)$ have at least 2 elements. Since Φ_L preserves the cardinality of generic fibers in V , then it also preserves the cardinality of generic fibers in $V \cap (P_1 \times P_2)$. This implies that the fiber in $\Phi_L(V \cap (P_1 \times P_2))$ over a is one-element, and generic fibers in $\Phi_L(V \cap (P_1 \times P_2))$ over P_1 have at least 2 elements. Hence, $\Phi_L(a, b) \in \mathcal{S}(\Phi_L(V \cap (P_1 \times P_2))) \subset \mathcal{S}(\Phi_L(V))$.

Finally, for any analytic subset X of an open set $\tilde{U} \subset \mathbf{C}^m$ let $X_{(k)} \subset \tilde{U}$ denote the union of all analytic irreducible components of X of dimension k .

3 Approximation

3.1 The main theorem

Theorem 3.1 *Let U be an open subset of \mathbf{C}^n and let $F : U \rightarrow \mathbf{C}^{\hat{m}}$ be a holomorphic map that satisfies a system of equations $Q(F(x)) = 0$ for $x \in U$, where $Q : \mathbf{C}^{\hat{m}} \rightarrow \mathbf{C}^q$ is a polynomial map. Then for every $x_0 \in U$ there are an open neighborhood $U_0 \subset U$ and a sequence $\{F^\nu : U_0 \rightarrow \mathbf{C}^{\hat{m}}\}$ of Nash maps converging uniformly to $F|_{U_0}$ such that $Q(F^\nu(x)) = 0$ for every $x \in U_0$ and $\nu \in \mathbf{N}$.*

As said in Section 1, this theorem is known. The proof given in the present paper is simpler than the previous ones and it allows us to design an algorithm for computing the approximating maps.

The proof will be divided into two parts. In part 1, we shall show how to reduce the problem to the case (c1) specified as follows:

- (c1) $q = 1$, and $Q : \mathbf{C}^{\hat{m}} \approx \mathbf{C}_u^m \times \mathbf{C}_z \rightarrow \mathbf{C}$ is a monic polynomial in z such that $R_Q(f_1, \dots, f_m) \neq 0$, where $R_Q \in \mathbf{C}[u]$ is the discriminant of Q as a polynomial in z , whereas f_1, \dots, f_m are the first m components of F .

In part 2, we shall show that given Q, F (as in Theorem 3.1) such that (c1) holds, we can produce a holomorphic map g depending on $n - 1$ variables and a polynomial map T with $T \circ g = 0$ such that if g can be locally approximated by Nash maps g^ν with $T \circ g^\nu = 0$, then F can be locally approximated by Nash maps F^ν with $Q \circ F^\nu = 0$. For $n = 1$, g will be a constant vector and we shall take $g^\nu = g$. Once parts 1, 2 are completed, the proof of Theorem 3.1 will be completed as well (by induction on n).

The following lemma will be useful in part 1 of the proof. Let U be a domain in \mathbf{C}^n . (For the notion of an optimal polynomial see Section 2.2.)

Lemma 3.2 *Assume we are given:*

(1) *a holomorphic map $(f_1, \dots, f_m, f_{m+1}, \dots, f_{m+s}) : U \rightarrow V$, where $V \subset \mathbf{C}^m \times \mathbf{C}^s$ is an algebraic variety of pure dimension m with proper projection onto \mathbf{C}^m ,*

(2) *a \mathbf{C} -linear map $L : \mathbf{C}^s \rightarrow \mathbf{C}$, $L \neq 0$, such that the generic fiber in V over \mathbf{C}^m and the generic fiber in $\Phi_L(V) \subset \mathbf{C}^m \times \mathbf{C}$ over \mathbf{C}^m have the same cardinality, where $\Phi_L : \mathbf{C}_u^m \times \mathbf{C}_v^s \rightarrow \mathbf{C}_u^m \times \mathbf{C}_z$ is defined by $\Phi_L(u, v) = (u, L(v))$.*

Assume that $R_L(f_1, \dots, f_m) \neq 0$, where $R_L \in \mathbf{C}[u]$ is the discriminant of the optimal polynomial $P_L \in (\mathbf{C}[u])[z]$ with $P_L^{-1}(0) = \Phi_L(V)$. Then for all sequences $\{f_1^\nu\}, \dots, \{f_m^\nu\}, \{\tilde{f}^\nu\}$ of functions, holomorphic on U , converging locally uniformly to $f_1, \dots, f_m, L(f_{m+1}, \dots, f_{m+s})$, respectively, such that

$$(3.1) \quad P_L(f_1^\nu, \dots, f_m^\nu, \tilde{f}^\nu) = 0, \text{ for almost all } \nu \in \mathbf{N},$$

the following holds. There exist sequences $\{f_{m+1}^\nu\}, \dots, \{f_{m+s}^\nu\}$ of functions, holomorphic on U , converging locally uniformly to f_{m+1}, \dots, f_{m+s} , respectively, such that the image of the map $(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$ is contained in V for almost all $\nu \in \mathbf{N}$.

Proof of Lemma 3.2. Recall that $\Phi_L(V)$ is algebraic (cf. Section 2.2). Since V has proper projection onto \mathbf{C}^m , $\Phi_L(V)$ also has proper projection onto \mathbf{C}^m .

For any holomorphic mapping $H : E \rightarrow \mathbf{C}^m$, where E is an open subset of \mathbf{C}^n , and any algebraic subvariety X of $\mathbf{C}^m \times \mathbf{C}^s$, consider the analytic set

$$\mathcal{V}(X, H) = \{(x, v) \in E \times \mathbf{C}^s : (H(x), v) \in X\}.$$

If X has pure dimension m and proper projection onto \mathbf{C}^m , then $\mathcal{V}(X, H)$ has pure dimension n and proper projection onto E . Indeed, fix any compact $K \subset E$. Then $H(K) \subset \mathbf{C}^m$ is also compact. Since X has proper projection onto \mathbf{C}^m , the fibers in X over $H(K)$ are uniformly bounded. Consequently, by definition of $\mathcal{V}(X, H)$, the fibers in $\mathcal{V}(X, H)$ over K are also uniformly bounded. Thus $\mathcal{V}(X, H)$ has proper projection onto E .

Now $\mathcal{V}(X, H)$ has dimension n because its projection onto the open set $E \subset \mathbf{C}^n$ is proper and surjective. The latter is true because the projection of X onto \mathbf{C}^m is surjective, which in turn follows from the fact that X has dimension m and proper projection onto \mathbf{C}^m . It remains to check that $\mathcal{V}(X, H)$ is of pure dimension (i.e. none of its analytic irreducible components has dimension strictly smaller than n). Suppose that this is not true. Then there is an open polydisc $B \times C$ relatively compact in $E \times \mathbf{C}^s$ such that $\mathcal{V}(X, H) \cap (B \times C) \neq \emptyset$, $\mathcal{V}(X, H) \cap (\overline{B} \times \partial C) = \emptyset$ and $\dim(\mathcal{V}(X, H) \cap (B \times C)) < n$. Then, by definition of $\mathcal{V}(X, H)$, we have $X \cap (H(B) \times C) \neq \emptyset$ and $X \cap (H(\overline{B}) \times \partial C) = \emptyset$. This implies that for every compact $K \subset H(B)$ we have $(K \times \mathbf{C}^s) \cap X \cap (H(B) \times C) = (K \times \overline{C}) \cap X$, hence $(K \times \mathbf{C}^s) \cap X \cap (H(B) \times C)$ is a compact set, so the projection

of $X \cap (H(B) \times C)$ onto $H(B)$ is proper. Since X has pure dimension m and $H(B) \subset \mathbf{C}^m$, the latter projection is surjective. Consequently, by definition of $\mathcal{V}(X, H)$, the projection of $\mathcal{V}(X, H) \cap (B \times C)$ onto B is surjective which contradicts the fact that $\dim(\mathcal{V}(X, H) \cap (B \times C)) < n$.

Next put $\Psi_L(x, v) = (x, L(v))$ for any $x \in \mathbf{C}^n, v \in \mathbf{C}^s$. Assume the notation of Section 2.2. Then we have the following

Lemma 3.3 *Let $Z \subset E \times \mathbf{C}^s$ be an analytic subset of pure dimension n with proper projection onto a domain $E \subset \mathbf{C}^n$ such that the generic fiber in Z and the generic fiber in $\Psi_L(Z)$ over E have the same cardinality. Then, for every analytic irreducible component Σ of $\Psi_L(Z)$ there exists an analytic irreducible component Γ of Z with $\Psi_L(\Gamma) = \Sigma$ such that the generic fiber in Γ and the generic fiber in Σ over E have the same cardinality.*

Proof of Lemma 3.3. Let $\bigcup_{j=1}^l Z_j$ be the decomposition of Z into pairwise distinct analytic irreducible components. Then every $\Psi_L(Z_j)$ is irreducible. Let, μ, ν, μ_j, ν_j denote the cardinalities of the generic fibers in $Z, \Psi_L(Z), Z_j, \Psi_L(Z_j)$ over E , respectively. Clearly, $\mu = \mu_1 + \dots + \mu_l$ and $\mu_j \geq \nu_j$ for every j . Since $\Psi_L(Z) = \bigcup_{j=1}^l \Psi_L(Z_j)$, we have $\nu_1 + \dots + \nu_l \geq \nu$. Now, by the hypothesis, $\nu = \mu$. Hence, $\nu_j = \mu_j$ for every j , as required. ■

The lemma allows us to complete the proof of Lemma 3.2. Put $\tilde{F} = (f_1, \dots, f_m), \tilde{F}^\nu = (f_1^\nu, \dots, f_m^\nu), G = (f_{m+1}, \dots, f_{m+s})$. Note that, by the second and the third paragraph of the proof, $\mathcal{V}(V, \tilde{F})$ and $\mathcal{V}(V, \tilde{F}^\nu)$ have pure dimension n and proper projection onto U .

Next observe that the cardinalities of the generic fibers in $\Psi_L(\mathcal{V}(V, \tilde{F}))$, $\mathcal{V}(V, \tilde{F})$, $\Psi_L(\mathcal{V}(V, \tilde{F}^\nu))$ and in $\mathcal{V}(V, \tilde{F}^\nu)$ over U are equal for large ν . Indeed, since $R_L \circ \tilde{F} \neq 0$, we also have $R_L \circ \tilde{F}^\nu \neq 0$, for large ν , and therefore the cardinalities of the generic fibers in $\mathcal{V}(V, \tilde{F})$ and in $\mathcal{V}(V, \tilde{F}^\nu)$ over U are equal. Now, by assumption (2) of Lemma 3.2 and by the fact that $R_L \circ \tilde{F} \neq 0$, the cardinalities of the generic fibers in $\Psi_L(\mathcal{V}(V, \tilde{F}))$ and in $\mathcal{V}(V, \tilde{F})$ over U are equal. Clearly, the latter is also true for $\Psi_L(\mathcal{V}(V, \tilde{F}^\nu))$ and $\mathcal{V}(V, \tilde{F}^\nu)$ for ν large enough.

In view of (3.1) we have $\text{graph}(\tilde{f}^\nu) \subset \Psi_L(\mathcal{V}(V, \tilde{F}^\nu))$. Hence, by Lemma 3.3 with $Z = \mathcal{V}(V, \tilde{F}^\nu)$, there is an analytic irreducible component Γ^ν of $\mathcal{V}(V, \tilde{F}^\nu)$ with $\Psi_L(\Gamma^\nu) = \text{graph}(\tilde{f}^\nu)$ such that the generic fiber in Γ^ν over U is one-element. Therefore, by (1), (2), (3) of Section 2.2, $\mathcal{S}(\Gamma^\nu) = \emptyset$ and $\Gamma^\nu = \text{graph}(G^\nu)$ for some holomorphic map $G^\nu : U \rightarrow \mathbf{C}^s$.

Let us show that $\{G^\nu\}$ converges to G locally uniformly. This will be done in three steps. First, we will show that $\{G^\nu\}$ is locally uniformly bounded. Next we will prove that $\{G^\nu\}$ converges to G pointwise. Finally, we will recall the proof of the fact that the local uniform convergence follows from the first two steps (cf. also [40]).

Observe that $\{G^\nu\}$ is locally uniformly bounded. Indeed, by $\Gamma^\nu \subset \mathcal{V}(V, \tilde{F}^\nu)$, the image of (\tilde{F}^ν, G^ν) is contained in V . Now, on one hand, $\{\tilde{F}^\nu\}$ is locally uniformly bounded (because it is a convergent sequence of holomorphic functions). On the other hand, $G^\nu(x)$ is contained in the fiber of V over $\tilde{F}^\nu(x)$, for every

$x \in U$. Since V has proper projection onto \mathbf{C}^m , the sequence $\{G^\nu\}$ is also locally uniformly bounded.

Now suppose that there is $x_0 \in U$ such that $\{G^\nu(x_0)\}$ does not converge to $G(x_0)$. Since $\{G^\nu(x_0)\}$ is bounded, there exists a subsequence $\{G^{\nu_i}(x_0)\}$ converging to some $b \neq G(x_0)$. Let $K \subset U$ be a compact ball with $\text{int}(K) \neq \emptyset$, $x_0 \in K$. By Montel's Theorem (cf. [20], p. 17) and by the choice of x_0 we may assume that $\{G^{\nu_i}\}$ converges uniformly on K to some $\tilde{G} \neq G|_K$ with $\text{graph}(\tilde{G}) \subset \mathcal{V}(V, \tilde{F})$. Now, $\Psi_L(\mathcal{V}(V, \tilde{F})) \cap (\{x\} \times \mathbf{C})$ and $\mathcal{V}(V, \tilde{F}) \cap (\{x\} \times \mathbf{C}^s)$ have equal cardinality for generic $x \in U$, so we can pick $x = x_1$ for which these sets do have equal cardinality and moreover, $\tilde{G}(x_1) \neq G(x_1)$. On one hand,

$$\Psi_L|_{\mathcal{V}(V, \tilde{F}) \cap (\{x_1\} \times \mathbf{C}^s)} : \mathcal{V}(V, \tilde{F}) \cap (\{x_1\} \times \mathbf{C}^s) \rightarrow \Psi_L(\mathcal{V}(V, \tilde{F})) \cap (\{x_1\} \times \mathbf{C})$$

is injective, because it is surjective and its domain and range are finite and have equal cardinality. On the other hand, $L \circ \tilde{G} = L \circ G$ because $L \circ G^{\nu_i} = \tilde{f}^{\nu_i}$, so $L(\tilde{G}(x_1)) = L(G(x_1))$ which contradicts the injectivity of $\Psi_L|_{\mathcal{V}(V, \tilde{F}) \cap (\{x_1\} \times \mathbf{C}^s)}$. Thus $\{G^\nu\}$ indeed converges to G pointwise.

If $\{G^\nu\}$ does not converge to G locally uniformly, then there are $\varepsilon > 0$, a subsequence $\{G^{\nu_j}\}$ and a sequence $\{x_{\nu_j}\}$ contained in some compact subset $K \subset U$ such that $\|G^{\nu_j}(x_{\nu_j}) - G(x_{\nu_j})\| > \varepsilon$ for every j . Then from $\{G^{\nu_j}\}$ one cannot choose a subsequence converging uniformly on K which, in view of the facts that $\{G^{\nu_j}\}$ is locally uniformly bounded and pointwise convergent, contradicts Montel's Theorem.

Now we can define f_{m+i}^ν to be the i 'th component of the map G^ν , for $i = 1, \dots, s$. ■

Proof of Theorem 3.1. Part 1. First, following [5], we carry out some preparations. Since the problem is local, it is sufficient to consider the case where U is connected. Then $F(U)$ is contained in an irreducible component of the algebraic variety $V = \{y \in \mathbf{C}^{\hat{m}} : Q(y) = 0\}$, so we may assume that V is of pure dimension, say m . We may also assume that $V \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ is with proper projection onto \mathbf{C}^m . Indeed, for a generic \mathbf{C} -linear isomorphism $J : \mathbf{C}^{m+s} \rightarrow \mathbf{C}^{m+s}$ the image $J(V)$ is with proper projection onto \mathbf{C}^m . Thus if there exists a sequence $H^\nu : U_0 \rightarrow J(V)$ of Nash mappings converging to $J \circ F|_{U_0}$ then the sequence $\{J^{-1} \circ H^\nu\}$ satisfies the assertion of the theorem.

Now the problem will be reduced to the case where V is a hypersurface (see [5], compare also [34], p. 394). Let $L : \mathbf{C}^s \rightarrow \mathbf{C}$ be any \mathbf{C} -linear form such that the generic fibers in $\Phi_L(V)$ and in V over \mathbf{C}^m have the same cardinality, where $\Phi_L : \mathbf{C}_u^m \times \mathbf{C}_v^s \rightarrow \mathbf{C}_u^m \times \mathbf{C}_z$ is defined by $\Phi_L(u, v) = (u, L(v))$. (The generic L has this property. It is clear that $\Phi_L(V)$ is an algebraic hypersurface of $\mathbf{C}_u^m \times \mathbf{C}_z$ with proper projection onto \mathbf{C}_u^m .) Let $P_L \in (\mathbf{C}[u])[z]$ be the monic polynomial in z with nonzero discriminant $R_L \in \mathbf{C}[u]$ and with $P_L^{-1}(0) = \Phi_L(V)$. (Such P_L is called the optimal polynomial for $\Phi_L(V)$, cf. Section 2.2.)

Let $f_1, \dots, f_m, f_{m+1}, \dots, f_{m+s}$ denote the coordinates of F . Observe that we may assume $R_L(f_1, \dots, f_m) \neq 0$. Indeed, otherwise we return to the very beginning of part 1 and repeat the whole construction with V replaced by $V \cap \{R_L = 0\}$. Since the latter variety is of pure dimension $m - 1$, we eventually

reach the required condition. (The image of the projection of $V \cap \{R_L = 0\}$ onto \mathbf{C}^m is $\{R_L = 0\}$. For more information on discriminant hypersurfaces cf. [22].)

If, in a neighborhood of some point, there exist Nash approximations $f_1^\nu, \dots, f_m^\nu, \tilde{f}^\nu$ of $f_1, \dots, f_m, L(f_{m+1}, \dots, f_{m+s})$, respectively, such that

$$P_L(f_1^\nu, \dots, f_m^\nu, \tilde{f}^\nu) = 0,$$

then there exists a Nash approximation F^ν of F with image contained in V . This is because, by Lemma 3.2, there exist holomorphic functions $f_{m+1}^\nu, \dots, f_{m+s}^\nu$ approximating f_{m+1}, \dots, f_{m+s} , respectively, such that the image of the map $(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$ is contained in V . Hence, it is sufficient to observe that $f_{m+1}^\nu, \dots, f_{m+s}^\nu$ must be Nash functions and next to define $F^\nu = (f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$. But these functions are Nash because

$$T_i(f_1^\nu, \dots, f_m^\nu, f_{m+i}^\nu) = 0,$$

where $T_i \in (\mathbf{C}[u])[z]$ is the optimal polynomial for $\pi_i(V)$, where $\pi_i : \mathbf{C}^m \times \mathbf{C}_1 \times \dots \times \mathbf{C}_s \rightarrow \mathbf{C}^m \times \mathbf{C}_i$ is the natural projection, so $\text{graph}(f_{m+i}^\nu)$ is contained in a Nash hypersurface $\{(x, z) : T_i(f_1^\nu(x), \dots, f_m^\nu(x), z) = 0\}$, for $i = 1, \dots, s$ (cf. Corollary 2.4).

Since $R_L(f_1, \dots, f_m) \neq 0$, we have completed part 1, i.e. we have reduced the problem to the case where it is sufficient to consider $Q = P_L$ and approximate the map $(f_1, \dots, f_m, L(f_{m+1}, \dots, f_{m+s}))$.

Part 2. Assume that Q, F satisfy (c1) and fix $x_0 \in U$. Without loss of generality we assume that x_0 is the origin in \mathbf{C}^n . We shall produce a holomorphic map g depending on $n - 1$ variables and a polynomial map T with $T \circ g = 0$ such that if g can be locally approximated by Nash maps g^ν with $T \circ g^\nu = 0$, then, in some neighborhood of the origin, F can be approximated by Nash maps F^ν with $Q \circ F^\nu = 0$. (For $n = 1$, g will be a constant vector, and then we take $g^\nu = g$.) Once this is completed, the proof of Theorem 3.1 will be completed as well (by induction on n).

The reduction from n to $n - 1$ will be carried out in a similar way to [1] (by applying the Weierstrass Preparation Theorem). The Tougeron Implicit Functions Theorem, which often appears in the context, is replaced here, as in [34], by the following lemma.

Lemma 3.4 ([34], p. 393). *Let d, M be positive integers. There is $\varepsilon > 0$ such that for every compact polydisc K centered at 0 and for every $A = a_0 z^d + a_1 z^{d-1} + \dots + a_d \in \mathcal{O}(K)[z]$ with $\sup_{x \in K} |a_i(x)| < M$ where $i = 0, \dots, d$ and for all $\alpha, c \in \mathcal{O}(K)$ with $\sup_{x \in K} |\alpha(x)| < M$, $\sup_{x \in K} |c(x)| < \varepsilon$ such that $A(\alpha) = c \cdot (\frac{\partial A}{\partial z}(\alpha))^2$ the following holds: there is $b \in \mathcal{O}(K)$ with $A(b) = 0$, and $\text{ord}_0(b - \alpha) \geq \text{ord}_0 c \cdot \frac{\partial A}{\partial z}(\alpha)$, and*

$$\sup_{x \in K} |b(x) - \alpha(x)| \leq 2 \sup_{x \in K} |c(x)| \left(\frac{\partial A}{\partial z}(\alpha)(x) \right).$$

Remark 3.5 The constant ε in Lemma 3.4 depends only on M, d and can be effectively calculated (see [34], the proofs of Lemma 1.5 and Lemma 1.6, pp 392-393).

It is more convenient for us to keep the notation of part 1. Hence, instead of Q we write P_L . The discriminant of P_L will be denoted by R_L , and the components of F will be denoted by $f_1, \dots, f_m, \tilde{f}$. Set $\tilde{F} = (f_1, \dots, f_m)$.

Let us turn to constructing g, T . Since $R_L \circ \tilde{F} \neq 0$ and $P_L(\tilde{F}, \tilde{f}) = 0$, we have $\frac{\partial P_L}{\partial z}(\tilde{F}, \tilde{f}) \neq 0$. Let $x', x = (x', x_n)$ be the tuples of the coordinates in $\mathbf{C}^{n-1}, \mathbf{C}^n$, respectively. We may assume that $\frac{\partial P_L}{\partial z}(\tilde{F}(o, \cdot), \tilde{f}(o, \cdot))$ has a zero of finite order, say d , at $x_n = 0$, where o denotes the origin in \mathbf{C}^{n-1} . (Otherwise we apply a linear change of the coordinates in \mathbf{C}^n .)

By the Weierstrass Preparation Theorem, $\frac{\partial P_L}{\partial z}(\tilde{F}(x), \tilde{f}(x)) = \hat{H}(x)W(x)$ in some neighborhood of the origin, where \hat{H} is a non-vanishing holomorphic function, and

$$W(x) = x_n^d + x_n^{d-1}a_1(x') + \dots + a_d(x')$$

is the Weierstrass polynomial. (For every $l = 1, \dots, d$, the function a_l is holomorphic in some neighborhood of o , and $a_l(o) = 0$.)

Dividing f_j, \tilde{f} by W^2 , one obtains, for $j = 1, \dots, m$:

$$\begin{aligned} f_j(x) &= H_j(x)W(x)^2 + r_j(x), \\ \tilde{f}(x) &= \tilde{H}(x)W(x)^2 + \tilde{r}(x) \end{aligned}$$

in some neighborhood of the origin, where H_j, \tilde{H} are holomorphic functions, and

$$\begin{aligned} r_j(x) &= x_n^{2d-1}b_{j,0}(x') + x_n^{2d-2}b_{j,1}(x') + \dots + b_{j,2d-1}(x'), \\ \tilde{r}(x) &= x_n^{2d-1}c_0(x') + x_n^{2d-2}c_1(x') + \dots + c_{2d-1}(x') \end{aligned}$$

are polynomials with coefficients holomorphic in some neighborhood of o .

Replacing the coefficients

$$a_1, \dots, a_d, b_{j,0}, \dots, b_{j,2d-1}, c_0, \dots, c_{2d-1},$$

for all j , in W, r_j, \tilde{r} by new variables denoted by the same letters we obtain polynomials $\omega, \rho_j, \tilde{\rho}$. Define:

$$\phi_j = \chi_j \omega^2 + \rho_j, \quad \tilde{\phi} = \tilde{\chi} \omega^2 + \tilde{\rho},$$

for $j = 1, \dots, m$, where $\chi_j, \tilde{\chi}$ are new variables. Now divide $P_L(\phi_1, \dots, \phi_m, \tilde{\phi})$ by ω^2 (treated as a polynomial in x_n with polynomial coefficients) and divide $\frac{\partial P_L}{\partial z}(\phi_1, \dots, \phi_m, \tilde{\phi})$ by ω to obtain

$$(3.2) \quad P_L(\phi_1, \dots, \phi_m, \tilde{\phi}) = \tilde{W}\omega^2 + x_n^{2d-1}T_1 + x_n^{2d-2}T_2 + \dots + T_{2d},$$

$$(3.3) \quad \frac{\partial P_L}{\partial z}(\phi_1, \dots, \phi_m, \tilde{\phi}) = \tilde{S}\omega + x_n^{d-1}T_{2d+1} + x_n^{d-2}T_{2d+2} + \dots + T_{3d},$$

where $\tilde{W}, \tilde{S}, T_1, \dots, T_{3d}$ are polynomials such that T_1, \dots, T_{3d} depend only on the variables

$$a_1, \dots, a_d, b_{j,0}, \dots, b_{j,2d-1}, c_0, \dots, c_{2d-1}.$$

Let

$$a_1, \dots, a_d, b_{j,0}, \dots, b_{j,2d-1}, c_0, \dots, c_{2d-1},$$

for $j = 1, \dots, m$, now denote the holomorphic coefficients of $W, r_1, \dots, r_m, \tilde{r}$. The tuple consisting of all these coefficients will be denoted by g , and this is the map announced in the first paragraph of part 2. Set $T = (T_1, \dots, T_{3d})$.

By uniqueness of the Weierstrass Division Theorem we have $T \circ g = 0$. It remains to check that if g can be locally approximated by a Nash map g^ν with $T \circ g^\nu = 0$, then, in some neighborhood of the origin, F can be approximated by a Nash map F^ν with $P_L \circ F^\nu = 0$.

Let $\{g^\nu\}$ be a sequence of Nash maps approximating g uniformly in some neighborhood of o with $T \circ g^\nu = 0$. The components of g^ν will be denoted by

$$a_1^\nu, \dots, a_d^\nu, b_{j,0}^\nu, \dots, b_{j,2d-1}^\nu, c_0^\nu, \dots, c_{2d-1}^\nu,$$

where $j = 1, \dots, m$.

Define:

$$\begin{aligned} W_\nu(x) &= x_n^d + x_n^{d-1} a_1^\nu(x') + \dots + a_d^\nu(x'), \\ r_{j,\nu}(x) &= x_n^{2d-1} b_{j,0}^\nu(x') + x_n^{2d-2} b_{j,1}^\nu(x') + \dots + b_{j,2d-1}^\nu(x'), \\ \tilde{r}_\nu(x) &= x_n^{2d-1} c_0^\nu(x') + x_n^{2d-2} c_1^\nu(x') + \dots + c_{2d-1}^\nu(x'), \end{aligned}$$

and, for $j = 1, \dots, m$, define

$$\begin{aligned} f_j^\nu(x) &= H_j^\nu(x)(W_\nu(x))^2 + r_{j,\nu}(x), \\ \bar{f}^\nu(x) &= \tilde{H}^\nu(x)(W_\nu(x))^2 + \tilde{r}_\nu(x). \end{aligned}$$

Here $\{H_j^\nu\}, \{\tilde{H}^\nu\}$ are any sequences of polynomials converging uniformly to H_j, \tilde{H} , respectively, in some neighborhood of the origin.

Now it is easy to see that by (3.2), (3.3) and the way $f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu$ are defined, there is a neighborhood of $0 \in \mathbf{C}^n$ in which, for sufficiently large ν , the following holds:

$$P_L(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = C^\nu \cdot \left(\frac{\partial P_L}{\partial z}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) \right)^2,$$

where $\{C^\nu\}$ is a sequence of holomorphic functions converging to zero uniformly. Uniform convergence of $\{C^\nu\}$ requires a brief explanation. First, there is a closed polydisc $\bar{E}' \times \bar{E}'' \subset \mathbf{C}^{n-1} \times \mathbf{C}$ centered at 0 such that $\{f_1^\nu\}, \dots, \{f_m^\nu\}, \{\bar{f}^\nu\}$ converge uniformly on $\bar{E}' \times \bar{E}''$ and, for ν large enough,

$$(\bar{E}' \times \partial E'') \cap \{x : \frac{\partial P_L}{\partial z}(f_1^\nu(x), \dots, f_m^\nu(x), \bar{f}^\nu(x)) = 0\} = \emptyset.$$

Then $\{C^\nu\}$ converges to 0 uniformly on $\bar{E}' \times \partial E''$ so, by the Maximum Principle, it converges to 0 uniformly on $E' \times E''$.

In view of the previous paragraph, it suffices to apply Lemma 3.4 with $A = P_L(f_1^\nu, \dots, f_m^\nu, z)$, $\alpha = \bar{f}^\nu$ and $c = C^\nu$ (for sufficiently large ν) to obtain

$$P_L(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = 0,$$

in some neighborhood U_0 of the origin, where $\{\tilde{f}^\nu\}$ is a sequence of holomorphic functions converging to \tilde{f} in U_0 . Since f_1^ν, \dots, f_m^ν are Nash functions, \tilde{f}^ν is a Nash function as well (because its graph is contained in a Nash hypersurface $\{(x, z) : P_L(f_1^\nu(x), \dots, f_m^\nu(x), z) = 0\}$; cf. Corollary 2.4).

Defining $F^\nu = (f_1^\nu, \dots, f_m^\nu, \tilde{f}^\nu)$ we complete part 2. ■

3.2 Algorithms

3.2.1 Computing with holomorphic functions

In Section 3.2, we work under the assumption that given a complex number z we can tell whether $z = 0$ or not. The assumption is justified by the fact that in practice \mathbf{C} is replaced by a computable subfield K such that for $z \in K$ one can decide whether $z = 0$ or not.

In this subsection the model of computation for our algorithms is described. It is clear that not every holomorphic function can constitute (a part of) input data. Only objects which can be encoded as finite sequences of symbols can be considered. Therefore, we assume (cf. [36], [38]) that every function f depending on x_1, \dots, x_n is given by a finite procedure $Expand_f$ which for every tuple $(k_1, \dots, k_n) \in \mathbf{N}^n$ returns the coefficient of the monomial $x_1^{k_1} \dots x_n^{k_n}$ of the Taylor expansion of f around zero. The input data corresponding to the function f consist of the procedure $Expand_f$, the size of a polydisc U_f centered at zero and a constant M_f such that: $\sup_{x \in U_f} |f(x)| < M_f$ and the Taylor series of f is convergent on (some domain containing) U_f . Then we can control accuracy of polynomial approximation of f . More precisely, the Cauchy integral formula yields estimates (majorants) for the coefficients of the Taylor expansion of f . Using these estimates we can compute, for a given polydisc \tilde{U} relatively compact in U_f centered at zero and for $\varepsilon > 0$, an integer N such that $\sup_{x \in \tilde{U}} |\tilde{f}(x) - f(x)| < \varepsilon$, where \tilde{f} is the Taylor polynomial of f at zero of order N . (For more information on Taylor models, majorant series and accuracy control in various settings see [17], [26], [27], [28], [38], [39] and references therein.)

Observe that for functions represented in this way, it is possible to perform ring operations and compute differentials (cf. [35], [36] and references therein). Namely, having input data for two functions f, g one can recover $Expand_{f+g}$, $Expand_{f \cdot g}$, $Expand_{\frac{\partial f}{\partial x_i}}$, for every $i \in \{1, \dots, n\}$, and $Expand_{\frac{1}{f}}$, if $f(0) \neq 0$. One can also set $U_{f+g} = U_f \cap U_g$, $M_{f+g} = M_f + M_g$ and $U_{f \cdot g} = U_f \cap U_g$, $M_{f \cdot g} = M_f \cdot M_g$. Moreover, the Cauchy integral formula implies $\sup_{x \in \theta U_f} |\frac{\partial f}{\partial x_i}(x)| < \frac{M_f}{r_i(1-\theta)}$, for every $\theta \in (0, 1)$, where θU_f is the polydisc centered at the origin of polyradius $(\theta r_1, \dots, \theta r_n)$, where (r_1, \dots, r_n) is the polyradius of U_f . So one can define $U_{\frac{\partial f}{\partial x_i}} = \theta U_f$ and $M_{\frac{\partial f}{\partial x_i}} = \frac{M_f}{r_i(1-\theta)}$ for some θ . Finally, once we know the bounds for the partial derivatives of f , we can compute a polydisc $\tilde{U} \subset U_f$ such that $|f(x)| > \frac{1}{2}|f(0)|$, for every x in the closure of \tilde{U} , provided that $f(0) \neq 0$. Indeed, we can clearly compute a constant M such that, after shrinking U_f slightly, the derivatives $D_x f$ satisfy $\sup_{x \in U_f} \|D_x f\| < M$. Then $|f(x) - f(0)| \leq M\|x\|$ for $x \in U_f$, so $|f(x)| \geq |f(0)| - M\|x\|$. Then it is sufficient to choose the radius of

\tilde{U} so small that $|f(0)| - M\|x\| > \frac{|f(0)|}{2}$ for every x in the closure of \tilde{U} . Therefore we can set $U_{\frac{1}{f}} = \tilde{U}$, $M_{\frac{1}{f}} = \frac{2}{|f(0)|}$.

Let us recall that in this model we have effective versions of the Weierstrass Preparation and Division Theorems (which will be used in the algorithms). Since the former is a consequence of the latter (see [15], p. 319), it is sufficient to discuss the Division Theorem. Let f, g be functions (depending on x_1, \dots, x_n) holomorphic in some polydiscs U_f, U_g centered at the origin, such that $f(o, x_n) = x_n^d u(x_n)$, for some $d \in \mathbf{N}$, where u is holomorphic, $u(0) \neq 0$, and o is the origin in \mathbf{C}^{n-1} . Then, according to the Division Theorem, there are a holomorphic function q and a polynomial r in x_n with $\deg(r) < d$ and with holomorphic coefficients such that, in some neighborhood of the origin, $g = qf + r$. We need to show that given data representing f, g , we can recover the data representing q, r . The existence of the procedures $Expand_q, Expand_r$ is a consequence of the proof of the Division Theorem (see [15], pp. 318-319; such procedures are in fact written, cf. [15] p. 544).

It remains to compute U_q, U_r, M_q, M_r . Since $f(o, x_n) = x_n^d u(x_n)$, $u(0) \neq 0$, there are a polydisc $A \times B \subset \mathbf{C}_{x_1, \dots, x_{n-1}}^{n-1} \times \mathbf{C}_{x_n}$ centered at the origin and a constant c with $\inf_{\overline{A} \times \partial B} |f| > c$. Such A, B, c can be computed by means of $Expand_f, U_f, M_f$. Indeed, as above, we can compute a disc $B_t \subset \mathbf{C}$ (centered at zero of radius t) such that $|u(x_n)| \geq \frac{|u(0)|}{2}$ on B_t and, moreover, a polydisc $A_s \subset \mathbf{C}^{n-1}$ (centered at zero of radius s) such that $A_s \times B_t$ is relatively compact in U_f . Then $|f(o, x_n)| \geq t^d \frac{|u(0)|}{2}$ on ∂B_t . Since we know the bounds for the partial derivatives of f on $A_s \times B_t$, we can compute M such that for any $\tilde{s} \leq s$ and any $(x, y) \in A_{\tilde{s}} \times \partial B_t$ we have $|f(x, y) - f(o, y)| \leq M\tilde{s}$. Then $|f(x, y)| \geq t^d \frac{|u(0)|}{2} - M\tilde{s}$, so we can take $B = B_t$, $A = A_{\tilde{s}}$, $c = t^d \frac{|u(0)|}{2} - M\tilde{s}$ for \tilde{s} small enough.

Observe that $\overline{A \times B}$ may be additionally assumed to be contained in $U_f \cap U_g$. Let $C \subset B \subset \mathbf{C}$ be another disc centered at zero of radius, say, $\frac{1}{2}$ of the radius of B , and let $U_q = U_r = A \times C$. Then the Taylor series of q, r at zero are convergent on U_q . Now, by the integral representations of q, r (see [25], p. 112), for $(x', x_n) \in A \times C$, where $x' = (x_1, \dots, x_{n-1})$, we have

$$q(x', x_n) = \frac{1}{2\pi i} \int_{\partial B} \frac{g(x', s)}{f(x', s)} \frac{1}{s - x_n} ds,$$

$$r(x', x_n) = \frac{1}{2\pi i} \int_{\partial B} \frac{g(x', s)}{f(x', s)} \frac{f(x', s) - f(x', x_n)}{s - x_n} ds.$$

Since $|g|, |f|$ are bounded from above on $\overline{A \times B}$ by M_g, M_f , and $|f|$ is bounded from below on $A \times \partial B$ by c , and $|s - x_n|$ is bounded from below for $x_n \in C, s \in \partial B$ by $\frac{1}{2}$ of the radius of B , we easily obtain $M_q = \frac{2M_g}{c}$, $M_r = \frac{4M_g M_f}{c}$.

In such a general model as the one considered in the present paper we cannot effectively decide whether a given holomorphic function equals zero identically. That is, the zero-test is not computable. It is only semi-computable. Namely, we can compute the coefficients of the Taylor expansion of higher and higher

orders to check whether some of them are non-zero. If we come across a non-zero one then the function is non-zero. But if the function equals zero, then the procedure does not stop. (Here let us mention that the zero-test is computable for certain subclasses of our class of functions. For details see [37] and references therein.) In particular, given a holomorphic map f and an algebraic set V , we cannot decide whether the image of f is contained in V . On the other hand, we have the following

Remark 3.6 *Let $V_1, V_2 \subset \mathbf{C}^m$ be algebraic subsets and let U be an open connected neighborhood of $0 \in \mathbf{C}^n$. Let $f = (f_1, \dots, f_m) : U \rightarrow \mathbf{C}^m$ be a holomorphic map, where f_1, \dots, f_m are represented in the data structure introduced above. Assume $f(U) \not\subseteq V_1 \cap V_2$. Then we can effectively point at (at least one) $i \in \{1, 2\}$ such that $f(U) \not\subseteq V_i$.*

We shall use this fact in the sequel. To see that it is true, note that at least one of the finite number of polynomials defining V_1 or of polynomials defining V_2 is a non-zero function after being composed with f . If we know that in a finite family of functions (at least) one of the functions does not equal zero identically, then, clearly, detecting such a function is effectively computable.

In the next subsection, we will present a partial algorithm relying on the **zero-test** which, as mentioned above, is not computable. This problem can be partially handled as follows. Given a holomorphic function \hat{R} and $\theta \in (0, 1)$, for every $\varepsilon > 0$ we can compute an integer $N(\theta, \varepsilon, \hat{R})$ such that the following holds. If the coefficients of the monomials of order smaller than $N(\theta, \varepsilon, \hat{R})$ (of the Taylor expansion of \hat{R}) all equal zero, then $\sup_{\theta U_{\hat{R}}} |\hat{R}| < \varepsilon$. (Recall that we have assumed that for $z \in \mathbf{C}$ we can decide whether $z = 0$ or not.) The number $N(\theta, \varepsilon, \hat{R})$ can be computed in the following way. Let $\hat{R} = \sum_{\alpha \in \mathbf{N}^n} a_\alpha z^\alpha$ be the Taylor expansion of \hat{R} . Suppose that for some $N(\theta, \varepsilon, \hat{R})$ the coefficients $a_\alpha = 0$ for every $|\alpha| \leq N(\theta, \varepsilon, \hat{R})$. Then by the Cauchy inequalities we have

$$\sup_{\theta U_{\hat{R}}} |\hat{R}| < M_{\hat{R}} \sum_{|\alpha| > N(\theta, \varepsilon, \hat{R})} \theta^{|\alpha|}.$$

Now for every $\varepsilon > 0$ we can compute how large $N(\theta, \varepsilon, \hat{R})$ **has to** be to ensure that the righthand side of the last inequality is smaller than ε .

This implies that after computing a finite number of the coefficients of the Taylor expansion of \hat{R} we can decide whether:

- (i) $\hat{R} \neq 0$ (if at least one of these coefficients is non-zero)
- (ii) $\sup_{\theta U_{\hat{R}}} |\hat{R}| < \varepsilon$ (else).

Therefore we assume $\hat{R} = 0$ if the coefficients of the monomials of order smaller than $N(\theta, \varepsilon, \hat{R})$ all equal zero (i.e. we assume $\hat{R} = 0$ if (ii) holds), where ε is some fixed very small positive real number. It may happen that (ii) holds and $\hat{R} \neq 0$, and then the assumption $\hat{R} = 0$ may give rise to a non-correct output of the algorithm. A complete solution, obtained by refining the ideas presented below, will be given in Section 3.2.3, where the zero-test will be avoided.

3.2.2 Partial algorithm

This subsection is devoted to a recursive algorithm of Nash approximation of a holomorphic mapping $F : U \rightarrow V \subset \mathbf{C}^{\hat{m}}$, where U is a neighborhood of zero in \mathbf{C}^n , for any $n \in \mathbf{N}$, and V is an algebraic variety. We will first state the algorithm and then comment on the main steps. The algorithm relies on zero-test, which is semi-computable in our model of computation. For this reason, for some input it may not return correct output data (and that is why we call it partial). This occurs when the image of F is very close to the singular locus $\text{Sing}(V)$ of the target variety but not contained in $\text{Sing}(V)$. If every zero-test returns the negative answer, which is usually the case, then the output we obtain is correct. In Section 3.2.3 we will show how to avoid the zero-test, which will be done by refining the algorithm discussed below.

Components of F will be represented as described in Section 3.2.1.

Input: a positive integer ν , an algebraic variety V , and a holomorphic mapping $F = (f_1, \dots, f_{\hat{m}}) : U \rightarrow V \subset \mathbf{C}^{\hat{m}}$, where U is an open connected neighborhood of $0 \in \mathbf{C}_x^n$.

Output: $P_i^\nu(x, z_i) \in (\mathbf{C}[x])[z_i]$, $P_i^\nu \neq 0$ for $i = 1, \dots, \hat{m}$, with the following properties:

- (a) $P_i^\nu(x, f_i^\nu(x)) = 0$ for every $x \in U_0$, where $F^\nu = (f_1^\nu, \dots, f_{\hat{m}}^\nu) : U_0 \rightarrow V$ is a holomorphic mapping such that $\|F - F^\nu\|_{U_0} < \frac{1}{\nu}$ and U_0 is an open neighborhood of $0 \in \mathbf{C}^n$ independent of ν ,
- (b) P_i^ν is a monic polynomial in z_i of degree in z_i bounded by a constant independent of ν .

Before discussing the problem in general, let us look at the special case where V is of pure dimension m for some $m > 0$ (if $m = 0$, then F is constant) and $F(0) \in \text{Reg}(V)$. Then, after a generic linear change of the coordinates, $V \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ has proper projection onto \mathbf{C}^m . Moreover, generic fibers in V and in $\Phi(V)$ over \mathbf{C}^m have the same cardinalities, where $\Phi(z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s}) = (z_1, \dots, z_m, z_{m+1})$. Furthermore, the fiber in $\Phi(V)$ over $(f_1, \dots, f_m)(0) \in \mathbf{C}^m$ has the same number of elements as the generic fiber in $\Phi(V)$ over \mathbf{C}^m .

Now calculate the optimal polynomial $P \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ describing $\Phi(V) \subset \mathbf{C}^m \times \mathbf{C}$ and the discriminant R of P . By the previous paragraph, $R(f_1, \dots, f_m)$ and $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})$ do not vanish in some neighborhood of $0 \in \mathbf{C}^n$. Moreover, $P(f_1, \dots, f_{m+1}) = 0$, so for Taylor polynomials $f_1^\nu, \dots, f_m^\nu, \bar{f}_{m+1}^\nu$ close to f_1, \dots, f_m, f_{m+1} , respectively, we have a holomorphic function

$$C^\nu(x) = \frac{P(f_1^\nu(x), \dots, f_m^\nu(x), \bar{f}_{m+1}^\nu(x))}{\left(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu(x), \dots, f_m^\nu(x), \bar{f}_{m+1}^\nu(x))\right)^2}$$

such that $|C^\nu(x)|$ is small. Consequently, by [Lemma 3.4](#), there is a Nash function f_{m+1}^ν close to f_{m+1} such that $P(f_1^\nu, \dots, f_{m+1}^\nu) = 0$. Using [Remark 3.5](#) we can estimate how close f_1^ν, \dots, f_m^ν to f_1, \dots, f_m **have to** be to ensure that f_{m+1}^ν approximates f_{m+1} with the required precision on some neighborhood of $0 \in \mathbf{C}^n$.

For f_1^ν, \dots, f_m^ν close enough to f_1, \dots, f_m we also have $R(f_1^\nu, \dots, f_m^\nu) \neq 0$. Then, there are holomorphic functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ such that the image of $(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$ is contained in V . Indeed, set $H = (f_1^\nu, \dots, f_m^\nu)$, $z = (z_{m+1}, \dots, z_{m+s})$, $\mathcal{V}(V, H) = \{(x, z) : (H(x), z) \in V\}$, $L(z) = z_{m+1}$, and $\Psi_L(x, z) = (x, L(z))$ and confirm (previous paragraph) that

$$\Psi_L(\mathcal{V}(V, H)) = \{(x, z_{m+1}) : P(f_1^\nu(x), \dots, f_m^\nu(x), z_{m+1}) = 0\}.$$

(For properties of $\mathcal{V}(V, H)$ cf. the discussion preceding Lemma 3.3.) Since $\Psi_L(\mathcal{V}(V, H))$ contains the graph of f_{m+1}^ν , then, by Lemma 3.3 with $Z = \mathcal{V}(V, H)$, we have that $\mathcal{V}(V, H)$ contains the graph of $(f_{m+1}^\nu, f_{m+2}^\nu, \dots, f_{m+s}^\nu)$ for some holomorphic functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$.

Now there are $Q_2, \dots, Q_s \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ such that

$$V \setminus R^{-1}(0) = \{R \neq 0, P = 0, z_{m+j}R = Q_j, j = 2, \dots, s\}.$$

(The polynomials Q_2, \dots, Q_s can be effectively computed, cf. [19] p. 233, [16], [21].) Consequently, the functions $f_1^\nu, \dots, f_{m+s}^\nu$ satisfy the equations

$$f_{m+j}^\nu R(f_1^\nu, \dots, f_m^\nu) = Q_j(f_1^\nu, \dots, f_{m+1}^\nu), \text{ for } j = 2, \dots, s$$

(which in particular implies that $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ are not only holomorphic but Nash as well).

Since the functions f_1, \dots, f_{m+s} also satisfy these equations, we can estimate (given Q_j, R) how close $f_1^\nu, \dots, f_{m+1}^\nu$ to f_1, \dots, f_{m+1} **have to** be to ensure that $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ approximate f_{m+2}, \dots, f_{m+s} with the required precision. When we know that the approximation exists, the polynomials P_i^ν can be computed as follows.

For $i = 1, \dots, m$, set $P_i^\nu(x, z_i) = z_i - f_i^\nu(x)$. Next define

$$V^\nu = \{(x, z) \in \mathbf{C}_x^n \times \mathbf{C}_z^{m+s} : z \in V, z_i = f_i^\nu(x), \text{ for } i = 1, \dots, m\},$$

where $z = (z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s})$. Finally, for $i = 1, \dots, s$, take $P_{m+i}^\nu \in (\mathbf{C}[x])[z_{m+i}]$ to be the optimal polynomial describing the image of the projection of V^ν to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}$. (This projection is proper so the image is an algebraic variety.)

In the general case we have the following (partial) algorithm. Let us recall that $V_{(k)}$ denotes the union of all k -dimensional irreducible components of V .

Algorithm 1:

1. Replace V by $V_{(m)}$ such that $F(U) \subset V_{(m)}$.
2. After a generic linear change of the coordinates in $\mathbf{C}^{\hat{m}}$ we have:
 - (x) $V \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ has proper projection onto \mathbf{C}^m ,
 - (y) generic fibers in V and in $\Phi(V)$ over \mathbf{C}^m have the same cardinalities,

where $\Phi(z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s}) = (z_1, \dots, z_m, z_{m+1})$.

Apply such a change of the coordinates.
3. Calculate the optimal polynomial $P \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ describing $\Phi(V) \subset \mathbf{C}^m \times \mathbf{C}$. Calculate the discriminant $R \in \mathbf{C}[z_1, \dots, z_m]$ of P .

4. If $R(f_1, \dots, f_m) = 0$, then go to Step 2 with m, s, V replaced by $m - 1, s + 1, V \cap \{R = 0\}$, respectively. Otherwise $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1}) \neq 0$.
5. If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(0) \neq 0$, then $F(0) \in \text{Reg}(V)$ (this case has been discussed above). If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(0) = 0$, then apply a linear change of the coordinates in \mathbf{C}^n after which the following holds: $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})(x) = \hat{H}(x)W(x)$ in some neighborhood of $0 \in \mathbf{C}^n$, where \hat{H} is a holomorphic function, $\hat{H}(0) \neq 0$ and W is a monic polynomial in x_n with holomorphic coefficients vanishing at $0 \in \mathbf{C}^{n-1}$, depending on $x' = (x_1, \dots, x_{n-1})$. Put $d = \deg(W)$.
6. Divide f_i by $(W)^2$ to obtain $f_i = (W)^2 H_i + r_i$ in some neighborhood of $0 \in \mathbf{C}^n, i = 1, \dots, m+1$. Here H_i are holomorphic functions and r_i are polynomials in x_n with holomorphic coefficients depending on x' such that $\deg(r_i) < 2d$.
7. Treating $H_i, i = 1, \dots, m+1$, and all the coefficients of W, r_1, \dots, r_{m+1} as new variables (except for the leading coefficient 1 of W) apply the division procedure for polynomials to obtain:

$$\begin{aligned} P(W^2 H_1 + r_1, \dots, W^2 H_{m+1} + r_{m+1}) &= \tilde{W}W^2 + x_n^{2d-1}T_1 + x_n^{2d-2}T_2 + \dots + T_{2d}, \\ \frac{\partial P}{\partial z_{m+1}}(W^2 H_1 + r_1, \dots, W^2 H_{m+1} + r_{m+1}) &= \tilde{S}W + x_n^{d-1}T_{2d+1} + x_n^{d-2}T_{2d+2} + \dots + T_{3d}. \end{aligned}$$

Here T_1, \dots, T_{3d} are polynomials depending only on the variables standing for the coefficients of W, r_1, \dots, r_{m+1} . Moreover, $T_1 \circ g = \dots = T_{3d} \circ g = 0$, where g is the mapping whose components are all these coefficients (cf. Section 3.1).

8. If $n > 1$, then apply Algorithm 1 recursively with ν, F, V replaced by $\mu(\nu), g, \{T_1 = \dots = T_{3d} = 0\}$, respectively. (How to choose $\mu(\nu)$ is described below.) As a result, for every component $c(x')$ of $g(x')$ one obtains a monic polynomial $Q_c^\mu(x', t_c) \in (\mathbf{C}[x'])[t_c]$ which put in place of $P_i^\nu(x, z_i)$ satisfies (a) and (b) above with $\nu, x, z_i, f_i^\nu, F, F^\nu$ replaced by $\mu, x', t_c, c^\mu, g, g^\mu$, respectively. Here, for every c, c^μ is a Nash function approximating c , in some neighborhood of the origin in \mathbf{C}^{n-1} such that the mapping g^μ obtained by replacing every c of g by c^μ satisfies $T_1 \circ g^\mu = \dots = T_{3d} \circ g^\mu = 0$. If $n = 1$, then g is constant, $g = g^\mu$, and one gets the Q_c^μ 's immediately.

9. Approximate H_i for $i = 1, \dots, m$, by polynomials H_i^μ . (Accuracy of this approximation is discussed below.) Let $W_\mu, r_{1,\mu}, \dots, r_{m+1,\mu}$ be the polynomials in x_n defined by replacing the coefficients of W, r_1, \dots, r_{m+1} by their Nash approximations (i.e. the components of g^μ) determined in Step 8. Using Q_c^μ (for all c) and H_i^μ one can calculate $P_i^\nu \in (\mathbf{C}[x])[z_i]$, for $i = 1, \dots, m$, satisfying (b) and (a) with $f_i^\nu = H_i^\mu(W_\mu)^2 + r_{i,\mu}$ being the i 'th component of the mapping F^ν (whose last $\hat{m} - m$ components are determined by $P_{m+1}^\nu, \dots, P_{\hat{m}}^\nu$ obtained in the next step). To calculate P_i^ν , for $i = 1, \dots, m$, perform the ring operations in the construction of f_i^ν in their implicit representation (see Lemma 2.1).

10. Put $V^\nu = \{(x, z) \in \mathbf{C}_x^n \times \mathbf{C}_z^{m+s} : z \in V, P_i^\nu(x, z_i) = 0 \text{ for } i = 1, \dots, m\}$, where $z = (z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s})$. For $i = 1, \dots, s$, take $P_{m+i}^\nu \in (\mathbf{C}[x])[z_{m+i}]$ to be the optimal polynomial describing the image of the projection of V^ν to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}$.

Remark 3.7 *Observe that Step 4 can be replaced by:*

4'. If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1}) = 0$, then go to Step 2 with m, s, V replaced by $m-1, s+1, V \cap \{R=0\}$, respectively.

Note that $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1}) = 0$ implies $R(f_1, \dots, f_m) = 0$ but in general the converse is false.

In the remaining part of this section we comment on the main ideas of the algorithm. Then we prove that one can effectively control the error of approximation of F by F^ν . Finally, we show that the degree of P_i^ν in z_i is bounded by a constant independent of ν (i.e. independent of accuracy of approximation). The latter fact is important if one considers efficiency of performing ring operations for functions f_i^ν in their implicit representation.

The aim of Step 1 is to reduce the problem to the case where V is of pure dimension. Since a generic algebraic variety is irreducible, V is almost always purely dimensional, and Step 1 need not be performed. If V is not purely dimensional, then we proceed as follows. Since U is connected, there is an integer m such that $F(U) \subset V_{(m)}$. (Recall that $V_{(m)}$ denotes the union of all m -dimensional irreducible components of V .) To compute $V_{(m)}$ we first decompose V into equidimensional parts. (For algorithmic equidimensional decomposition see [14], p. 104, [15], p. 258, [18], [19], [23] and references therein, see also [30], [31].) Then it remains to check for a fixed m , whether $F(U)$ is contained in $V_{(m)}$. This is equivalent to testing whether $u_{m,l} \circ F$ is identically zero, for $l = 1, \dots, j_m$, where $u_{m,1}, \dots, u_{m,j_m}$ are polynomials describing $V_{(m)}$. (Here is the first appearance of the zero-test discussed in Section 3.2.1.) If we have the extra knowledge that $F(U)$ is not contained in $\text{Sing}(V)$, then the problem is decidable because there is precisely one m such that $F(U) \subset V_{(m)}$. (Then m can be found in a finite number of steps by excluding the other equidimensional components, cf. Remark 3.6.)

As for Step 2, the set of linear isomorphisms $\hat{J} : \mathbf{C}^m \times \mathbf{C}^s \rightarrow \mathbf{C}^m \times \mathbf{C}^s$ such that $\hat{J}(V)$ has proper projection onto \mathbf{C}^m is dense and open in the set of all linear maps. Moreover, given \hat{J} , one can test algorithmically whether $\hat{J}(V)$ does have proper projection onto \mathbf{C}^m . Thus, we can simply choose \hat{J} at random with very high probability and then check whether it is suitable. (This is discussed in detail in [19], p. 235.) Now if V has proper projection onto \mathbf{C}^m , then one can choose a linear form $L : \mathbf{C}^s \rightarrow \mathbf{C}$ such that generic fibers in $\Phi_L(V)$ and in V over \mathbf{C}^m have the same finite number of elements, where $\Phi_L(z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s}) = (z_1, \dots, z_m, L(z_{m+1}, \dots, z_{m+s}))$. For the problem of choosing L effectively, see [19], p. 233 or [16], Section 3.4.7, or [21], Proposition 27. For simplicity of notation we assume $L(z_{m+1}, \dots, z_{m+s}) = z_{m+1}$, which can be achieved by a linear change of coordinates in $\mathbf{C}^m \times \mathbf{C}^s$.

Once we have $\hat{J} : \mathbf{C}^m \times \mathbf{C}^s \rightarrow \mathbf{C}^m \times \mathbf{C}^s$ such that (x), (y) in Step 2 are satisfied with V replaced by $\hat{J}(V)$, we also replace F by $\hat{J} \circ F$. One thing, which requires explanation, concerns the fact that the output of the algorithm consists of hypersurfaces containing the graphs of approximating maps. Thus the question is how to recover the output for our original $F : U \rightarrow V$ if we

proceed with $\hat{J} \circ F : U \rightarrow \hat{J}(V)$ instead of F and obtain $\hat{F}_\nu : U_0 \rightarrow \hat{J}(V)$. Observe that the components of the Nash map $F_\nu = \hat{J}^{-1} \circ \hat{F}_\nu : U_0 \rightarrow V$ approximating $F|_{U_0}$ are linear combinations of the components of the map $\hat{F}_\nu : U_0 \rightarrow \hat{J}(V)$ approximating $\hat{J} \circ F$. This implies, by Lemma 2.1, that the output for F can be recovered from the output obtained for $\hat{J} \circ F$.

Set $F^* = (f_1, \dots, f_m, f_{m+1}) : U \rightarrow \Phi(V)$. Now the idea is to approximate the map $F^* : U \rightarrow \Phi(V)$ and then recover the output data for F having such data for F^* . To do this, we first need to calculate (Step 3) the optimal polynomial P for $\Phi(V)$ and the discriminant R of P . Computing the optimal polynomial P for $\Phi(V)$ (given V) is discussed in [19], p. 240-241 (where P is called the minimal polynomial). Let us note that the problem of effective elimination of variables (an instance of which is computing $\Phi(V)$) has been discussed in several works (cf. [4], [12], Chapter 3, or [15], pp. 69-73).

We need $R(f_1, \dots, f_m) \neq 0$, which can be achieved by repeating Steps 2-4 in a loop. More precisely, if in Step 4, we have $R(f_1, \dots, f_m) = 0$, then we can repeat Steps 2, 3 with m, s, V replaced by $m-1, s+1, V_1 = V \cap \{R=0\}$ (the new variety is purely $(m-1)$ -dimensional) and check the condition again. In this process, the dimension of the target variety drops, so finally the required condition is satisfied. (In Step 4 we have the zero-test discussed in Section 3.2.1.) Note that if R and the discriminant R^* of R vanish identically on the images of the corresponding maps, then after Steps 2, 3, 4 performed with $m-1, s+1, V_1$, the algorithm will return to Step 2 again. For such R , instead of performing Steps 2, 3 with $m-1, s+1, V_1$ (to obtain some new variety V_2 in Step 4) we can alternatively define $V_2 = V_1 \cap \{R^* = 0\}$ (then $\dim(V_2) = \dim(V_1) - 1$). If in addition R^* is reducible, then we can reduce the complexity of V_2 by taking (instead of R^*) a factor of R^* whose zero-set contains the image of the map. Here it is important that there are direct methods of computing factors of iterated discriminants (such as R^*) which in many cases are much more efficient than computing and factorizing iterated discriminants (for details see [22]). Let us make the alternative construction of V_2 more precise.

First observe that $V_R = \{(z_1, \dots, z_m) \in \mathbf{C}^m : R(z_1, \dots, z_m) = 0\}$ may be assumed to have proper projection onto $\mathbf{C}_{z_1, \dots, z_{m-1}}^{m-1}$. Then $V_1 = V \cap \{R=0\}$ has proper projection onto $\mathbf{C}_{z_1, \dots, z_{m-1}}^{m-1}$. Choose $R^* \in \mathbf{C}[z_1, \dots, z_{m-1}]$ such that $\{R^* = 0\} \subset \mathbf{C}^{m-1}$ is the set of points over which the fiber in V_R has not the maximal cardinality. If the discriminant of R is non-zero, then define R^* to be the discriminant of R . It is clear that if $R^*(f_1, \dots, f_{m-1}) = 0$, then the fibers in V_1 over the image of (f_1, \dots, f_{m-1}) have smaller cardinality than generic fibers in V_1 over \mathbf{C}^{m-1} . Hence, the discriminant R_1 that would be obtained if we performed Step 3 with V_1 (R_1 is related to V_1 in the same way as R is related to V) would vanish identically on the image of (f_1, \dots, f_{m-1}) . In this case, replace V_1 by $V_1 \cap \{S=0\}$ (of strictly smaller dimension), where S is a reduced factor of R^* vanishing on the image of (f_1, \dots, f_{m-1}) .

Finally note that if we know that the image of F is not contained in $\text{Sing}(V)$, then we need not replace V by a smaller variety in order to achieve $R(f_1, \dots, f_m) \neq 0$, because the generic linear change of variables in Step 2

gives this condition.

Now to approximate the map $F^*|_{U_0}$ by some Nash map $F^{*,\nu} : U_0 \rightarrow \Phi(V)$, where U_0 is some neighborhood of zero, we use Lemma 3.4 which implies that it is sufficient to find Nash functions $f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu$ approximating f_1, \dots, f_m, f_{m+1} , respectively, on U_0 such that

$$(3.4) \quad P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = C^\nu \left(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) \right)^2,$$

for some holomorphic function C^ν with small norm.

To find such functions we need to handle the zero-set of $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})$. (Recall that $R(f_1, \dots, f_m) \neq 0$, so $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1}) \neq 0$.) For that reason we change the coordinates in a neighborhood of zero in \mathbf{C}^n (Step 5) after which the Weierstrass Preparation Theorem can be applied to $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})$. More precisely, (f_1, \dots, f_{m+1}) may have to be replaced by $(f_1 \circ J, \dots, f_{m+1} \circ J)$, where $J : \mathbf{C}^n \rightarrow \mathbf{C}^n$ is a generic linear isomorphism. Since the coordinates are changed in the domain of the approximated map (and not in its range), the difficulties which we discussed in Step 2 do not appear here. In view of the Weierstrass Preparation Theorem (the effective version of which is discussed in Section 3.2.1), we assume that in some neighborhood of $0 \in \mathbf{C}^n$, the zero-set of the function $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})$ is given by the zero-set of a monic polynomial W in x_n with holomorphic coefficients depending on $x' = (x_1, \dots, x_{n-1})$, where x_1, \dots, x_n are the coordinates in \mathbf{C}^n .

The problem of finding $f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu$ satisfying the equation (3.4) can be solved using recursion. The aim of Steps 6 and 7 is to prepare the setup for this recursion. More precisely, we define a new map g depending only on the variables x_1, \dots, x_{n-1} whose image is contained in some algebraic variety and whose approximation enables us to obtain the solution to (3.4). To define g , we need the effective version of the Division Theorem (see Section 3.2.1). Step 8 is devoted to the recursive application of the algorithm.

The aim of Step 9 is to recover the solution to (3.4) from the data obtained by the recursive application of the algorithm. More precisely, we recover the functions f_1^ν, \dots, f_m^ν , which will constitute the first m components of the map F^ν approximating the original map F . This means that, for $i = 1, \dots, m$, we construct a polynomial $P_i^\nu(x, z_i)$ whose zero-set contains the graph of f_i^ν . As for the function \bar{f}^ν , it is sufficient to know that such a function satisfying (3.4) together with f_1^ν, \dots, f_m^ν exists. Here one can simply take $\bar{f}^\nu = H_{m+1}^\mu W_\mu^2 + r_{m+1,\mu}$, where H_{m+1}^μ is a polynomial approximation of H_{m+1} . Then, by Lemma 3.4, there exists a Nash approximation f_{m+1}^ν of f_{m+1} such that $P(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu) = 0$ which allows us to recover the output data for the remaining functions $f_{m+1}^\nu, \dots, f_{m+s}^\nu$ satisfying the required properties. How to construct these last s components of F^ν is discussed in Step 10.

Consider the algebraic variety V^ν defined in Step 10. Since $V \subset \mathbf{C}^m \times \mathbf{C}^s$ has proper projection onto \mathbf{C}^m and, for $i = 1, \dots, m$, $P_i^\nu(x, z_i)$ are monic in z_i , V^ν has proper projection onto \mathbf{C}_x^n . Hence, V^ν also has proper projection to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}^n$, for $i = 1, \dots, s$. Therefore the image $V^{\nu,i}$ of the projection of V^ν to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}^n$ is an algebraic hypersurface with proper projection onto \mathbf{C}_x^n .

Moreover, $V^{\nu,i}$ contains the graph of f_{m+i}^ν , for $i = 1, \dots, s$. Indeed, the fact that $V^{\nu,1}$ contains the graph of f_{m+1}^ν follows from $P(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu) = 0$. The last equation and Lemma 3.3 with $L(z_{m+1}, \dots, z_{m+s}) = z_{m+1}$ and $Z = \mathcal{V}(V, H)$ where $H = (f_1^\nu, \dots, f_m^\nu)$, imply (cf. the discussion preceding Algorithm 1) that there are holomorphic functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ such that the image of $(f_1^\nu, \dots, f_{m+s}^\nu)$ is contained in V . Hence, $V^{\nu,i}$ contains the graph of f_{m+i}^ν , for $i = 2, \dots, s$ (which implies that f_{m+i}^ν is Nash). Therefore it is sufficient to take P_{m+i}^ν to be the optimal polynomial for $V^{\nu,i}$, for $i = 1, \dots, s$.

It remains to prove that one can effectively control the error of approximation of F by F^ν on U_0 and that the degree of P_i^ν in z_i is bounded by a constant independent of ν .

3.2.2.1 Controlling accuracy of approximation

We will assume in Step 5 that $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(0) = 0$, since otherwise we have $F(0) \in \text{Reg}(V)$, which has been discussed already. It is sufficient to show that we can compute $\mu(\nu)$ (Step 8) and estimate the distance between H_i^μ and H_i (Step 9) which guarantee that $\|F - F^\nu\|_{U_0} < \frac{1}{\nu}$.

First, using the definition of f_i^ν (Step 9), for $i = 1, \dots, m$, we can determine how close H_i^μ, g^μ have to be to H_i, g to ensure that $\|f_i - f_i^\nu\|_{U_0} < \frac{1}{\nu}$.

As for f_{m+1}^ν , observe that, by the formulas in Step 7 and by Steps 8, 9 we have:

$$P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = C^\nu \left(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) \right)^2,$$

where $\bar{f}^\nu = H_{m+1}^\mu W_\mu^2 + r_{m+1, \mu}$, and H_{m+1}^μ is a polynomial approximation of H_{m+1} , and C^ν is a holomorphic function. More precisely, C^ν is the composition of \tilde{W}/\tilde{S}^2 with $g^\mu, H_1^\mu, \dots, H_{m+1}^\mu$. Observe that the composition of \tilde{S}^2 with g, H_1, \dots, H_{m+1} is non-vanishing in some neighborhood of the origin, whereas the composition of \tilde{W} with the same functions is identically zero. Moreover, in some neighborhood of every point at which \tilde{S} is non vanishing we have bounds for the partial derivatives of \tilde{W}/\tilde{S}^2 . Therefore we can compute **an open bounded polydisc U_1 in a neighborhood of $0 \in \mathbf{C}^n$ in which the algorithm is performed**, such that for every $\varepsilon > 0$ we can compute $\delta > 0$ such that if the distance between g^μ, H_i^μ and g, H_i is smaller than δ , then $\|C^\nu\|_{U_1} < \varepsilon$. For ε small enough, by Lemma 3.4 and Remark 3.5, there is a Nash function f_{m+1}^ν with $P(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu) = 0$ and $\|f_{m+1}^\nu - f_{m+1}\|_{U_1} < \frac{1}{\nu}$. It is clear that this is the f_{m+1}^ν whose graph is contained in the zero-locus of P_{m+1}^ν computed in Step 10.

As for f_{m+j}^ν , for $j = 2, \dots, s$, since $V \subset \mathbf{C}^m \times \mathbf{C}^s$ satisfies (x) and (y) (Step 2), there are effectively computable $Q_2, \dots, Q_s \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ such that

$$V \setminus R^{-1}(0) = \{R \neq 0, P = 0, z_{m+j}R = Q_j, j = 2, \dots, s\},$$

where $R \in \mathbf{C}[z_1, \dots, z_m]$ is the discriminant of the optimal polynomial $P \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ for $\Phi(V)$ (cf. [19] p. 233, [16], [21]).

Denote $\hat{R}(x) = R(f_1(x), \dots, f_m(x))$. After Step 4, $\hat{R} \neq 0$, so we may assume that $\hat{R}(o, \cdot)$ has a zero of finite order at $x_n = 0$, where $o \in \mathbf{C}^{n-1}$ is the origin.

Now we can compute polydiscs $A \subset \mathbf{C}^{n-1}, B \subset \mathbf{C}$, centered at zero, and a real number $c > 0$ such that $\overline{A \times B} \subset U_1$, and $\inf_{\overline{A \times \partial B}} |\hat{R}| > c$ (cf. the discussion on the Division Theorem in Section 3.2.1). We know how close g^μ, H_i^μ to g, H_i **have to** be to ensure that $\inf_{\overline{A \times \partial B}} |R(f_1^\nu, \dots, f_m^\nu)| > 0$. Then, once we have a Nash function f_{m+1}^ν with $P(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu) = 0$, by Lemma 3.3 with $L(z_{m+1}, \dots, z_{m+s}) = z_{m+1}$ and $Z = \mathcal{V}(V, H)$ where $H = (f_1^\nu, \dots, f_m^\nu)$, we obtain (cf. the discussion preceding Algorithm 1) that there are holomorphic functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ such that the image of $(f_1^\nu, \dots, f_{m+s}^\nu)$ is contained in V .

On (some neighborhood of) $\overline{A} \times \partial B$ we have, for $j = 2, \dots, s$,

$$f_{m+j} = \frac{Q_j(f_1, \dots, f_m, f_{m+1})}{R(f_1, \dots, f_m)}, \quad f_{m+j}^\nu = \frac{Q_j(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu)}{R(f_1^\nu, \dots, f_m^\nu)}.$$

Consequently, ($f_{m+2}^\nu, \dots, f_{m+s}^\nu$ are Nash and) we can compute $\delta > 0$ such that if $\|f_i - f_i^\nu\|_{A \times B} < \delta$, for $i = 1, \dots, m+1$, then $\|f_{m+j} - f_{m+j}^\nu\|_{\overline{A \times \partial B}} < \frac{1}{\nu}$ (hence, by the Maximum Principle, $\|f_{m+j} - f_{m+j}^\nu\|_{A \times B} < \frac{1}{\nu}$) for $j = 2, \dots, s$. But, as shown above, we know how large $\mu(\nu)$ **has to** be and how close H_i^μ to H_i **has to** be to ensure that $\|f_i - f_i^\nu\|_{A \times B} < \delta$, for $i = 1, \dots, m+1$. This means that we can control the error of approximation of F by F^ν on $A \times B$.

3.2.2.2 Bound for the degree of output polynomials

Finally, let us prove the claim: there is a bound for the degree of P_i^ν in z_i independent of ν . For any Nash function $h(x)$ defined in some open neighborhood of $0 \in \mathbf{C}^n$, by an implicit form of h we mean a non-zero polynomial $P_h(x, z_i)$ with $P_h(x, h(x)) = 0$ in some neighborhood of $0 \in \mathbf{C}^n$. By the degree of the implicit form $P_h(x, z_i)$ we mean the degree of the polynomial P_h in z_i .

We proceed by induction on n . If $n = 0$, then F is constant and there is nothing to prove. Let $n > 0$. Assume that the claim holds for $n - 1$. First let us consider P_i^ν for $i = 1, \dots, m$. Since H_i^μ , for $i = 1, \dots, m$, are polynomials, they have implicit forms of degree 1. The same is true for x_n^j , for $j \in \mathbf{N}$. By induction hypothesis, the coefficients of $W_\mu, r_{1,\mu}, \dots, r_{m,\mu}$ have implicit forms of degrees bounded by a constant independent of μ . Now recall that to calculate P_i^ν , for $i = 1, \dots, m$, we perform the ring operations in the construction of $f_i^\nu = H_i^\mu W_\mu^2 + r_{i,\mu}$ in their implicit representation. By Lemma 2.1, for Nash functions f, g with implicit forms P_f, P_g , we obtain implicit forms $P_{f+g}, P_{f \cdot g}$ whose degrees are bounded by a constant depending only on the degrees of implicit forms P_f, P_g . This shows our claim for $i = 1, \dots, m$.

By Step 10, for $i = 1, \dots, s$, P_{m+i}^ν is the optimal polynomial describing the image $V^{\nu,i}$ of the projection of V^ν to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}$. So the degree of P_{m+i}^ν in z_i equals the cardinality of generic fibers in $V^{\nu,i}$ over \mathbf{C}_x^n . By definition of V^ν , the latter number is bounded by a constant depending only on the cardinality of generic fibers in V over \mathbf{C}^m and on the degree of P_i^ν in z_i , for $i = 1, \dots, m$. In view of the previous paragraph, we obtain a bound independent of ν .

Remark 3.8 The main difference between Algorithm 1 and the method of approximation presented in [5] is that Algorithm 1 does not rely on factorization

of polynomials with holomorphic coefficients which appears in Step 7 of the method of [5]. The factorization is not computable in the model considered in the present paper. To see this, take any holomorphic function $a(x)$ and $W(x, y) = y(y - a(x))$. Then either $W = W_1^2$, where $W_1(x, y) = y$ (which occurs iff $a = 0$) or $W = W_1 \cdot W_2$, where $W_1(x, y) = y$ and $W_2(x, y) = y - a(x)$ (which occurs iff $a \neq 0$), hence, computability of the factorization would imply the decidability of the zero-test for a . In other words, given two very close (possibly equal) factors of W it is not possible to distinguish whether they are equal or not.

Step 7 (of the method of [5]) could be effectively performed if we knew that W (obtained in Step 5) does not have multiple factors. But W can have multiple factors (even if R obtained in Step 4 is replaced by a reduced polynomial). Indeed, let us consider any algebraic hypersurface $V \subset \mathbf{C}^m$ with $\text{Sing}(V) = \{0\}$ and any non-constant holomorphic map $f = (f_1, \dots, f_{\hat{m}}) : \mathbf{C}^2 \supset U \rightarrow V$, where U is an open connected neighborhood of $(0, 0)$ and, for $j = 1, \dots, \hat{m}$,

$$f_j(x_1, x_2) = g_j(x_1, x_2)u(x_1, x_2),$$

for some holomorphic g_j, u with $u(0, 0) = 0$. With these data let us try the method of [5] to compute approximation for f .

First we apply a generic linear change of the coordinates after which $V \subset \mathbf{C}_y^m \times \mathbf{C}$ has proper projection onto \mathbf{C}_y^m . After this change we also have that the discriminant $R(y)$ of the optimal polynomial for V satisfies $R(f_1, \dots, f_m) \neq 0$ (because $f(U) \not\subseteq \text{Sing}(V)$). Clearly,

$$R(f_1(x_1, x_2), \dots, f_m(x_1, x_2)) = G(x_1, x_2)u(x_1, x_2),$$

where $G \neq 0$ is holomorphic. (The latter formula remains true, maybe with some other G , after replacing R by the reduced polynomial having the same zero-set as R). Therefore, W obtained in Step 5 has multiple factors if u has multiple factors.

3.2.3 Complete algorithm

Here we improve Algorithm 1 to obtain a complete algorithm of approximation (without semi-computable steps). Every component of the approximated map will be represented as described in Section 3.2.1.

Input: a positive integer ν , an algebraic variety $V \subset \mathbf{C}^m$, and a holomorphic mapping $F = (f_1, \dots, f_{\hat{m}}) : U \rightarrow \mathbf{C}^m$, where U is an open connected neighborhood of $0 \in \mathbf{C}_x^n$.

The algorithm either detects that the image of F is not contained in V and then returns " $F(U) \not\subseteq V$ " or computes a Nash map $F^\nu : U_0 \rightarrow V$ such that $\|F^\nu - F\|_{U_0} < \frac{1}{\nu}$. Note that the problem of testing whether $F(U) \not\subseteq V$ is not decidable (it is semi-decidable), but one can perform at least one of two tasks stated above. In other words, the algorithm may compute an approximation of F into V even if $F(U) \not\subseteq V$ (but $F(U)$ is very close to V). The idea to

give the algorithm the choice "either detect or approximate" comes from the fact that the existence of a non-exact solution (f_1, \dots, f_{m+1}) of the equation $P(z_1, \dots, z_{m+1}) = 0$ satisfying (3.4) (with small C^ν) implies the existence of an exact solution (cf. Lemma 3.4). If $F(U) \subset V$, then the algorithm necessarily computes an approximation.

Output: Either " $F(U) \not\subset V$ " or $P_i^\nu(x, z_i) \in (\mathbf{C}[x])[z_i]$, $P_i^\nu \neq 0$ for $i = 1, \dots, \hat{m}$, with the following properties:

- (a) $P_i^\nu(x, f_i^\nu(x)) = 0$ for every $x \in U_0$, where $F^\nu = (f_1^\nu, \dots, f_m^\nu) : U_0 \rightarrow V$ is a holomorphic mapping such that $\|F - F^\nu\|_{U_0} < \frac{1}{\nu}$ and U_0 is an open neighborhood of $0 \in \mathbf{C}^n$ independent of ν ,
- (b) P_i^ν is a monic polynomial in z_i of degree in z_i bounded by a constant independent of ν .

Let us first discuss two simple cases to which the algorithm reduces the problem by recursive calls. Recall that we work under the assumption that the zero test can be performed in \mathbf{C} (cf. Section 3.2.1).

If $n = 0$ (i.e. F is a constant map), then we obtain output data immediately. Now observe that if V is zero-dimensional, then, for any n , we obtain output data almost immediately. Indeed, the algorithm works as follows: check whether $F(0) \in V$. If $F(0) \notin V$, then return " $F(U) \not\subset V$ ". Otherwise compute U_0 and $\kappa \in \mathbf{N}$ such that if all the coefficients (except for the terms of order zero) of the Taylor expansion of F at 0 up to order κ vanish, then $\|F - F(0)\|_{U_0} < \frac{1}{\nu}$. Check whether these coefficients vanish. If this is true, then the constant map $x \mapsto F(0)$ is the required approximation. Otherwise, F is a non-constant map defined on an open connected set so its image is not contained in any zero-dimensional variety (return " $F(U) \not\subset V$ ").

Denote $m = \dim(V)$. As mentioned above, the problem will be reduced to the case where $n \cdot m = 0$. More precisely, running with V, F , the algorithm will be called recursively either with the same map but with a target variety of strictly smaller dimension than m , or it will be called with some map depending on strictly fewer variables (and then the dimension of the target variety may even increase).

Before discussing the problem in general let us look at one more special case (where no recursive calls are necessary). Namely, assume that V is of pure dimension $m > 0$ and $F(0) \in \text{Reg}(V)$. Then, after a generic linear change of the coordinates, $V \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ has proper projection onto \mathbf{C}^m . Moreover, generic fibers in V and in $\Phi(V)$ over \mathbf{C}^m have the same cardinalities, where $\Phi(z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s}) = (z_1, \dots, z_m, z_{m+1})$. Furthermore, the fiber in $\Phi(V)$ over $(f_1, \dots, f_m)(0) \in \mathbf{C}^m$ has the same number of elements as generic fibers in $\Phi(V)$ over \mathbf{C}^m .

Now calculate the optimal polynomial $P \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ describing $\Phi(V) \subset \mathbf{C}^m \times \mathbf{C}$ and the discriminant R of P . By the previous paragraph, $R(f_1(0), \dots, f_m(0)) \neq 0 \neq \frac{\partial P}{\partial z_{m+1}}(f_1(0), \dots, f_{m+1}(0))$, so we can compute a polydisc E centered at $0 \in \mathbf{C}^n$ and $\tilde{c} > 0$ such that $\inf_E |R(f_1, \dots, f_m)| > \tilde{c}$ and

$\inf_E \left| \frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1}) \right| > \tilde{c}$, so

$$C(x) = \frac{P(f_1(x), \dots, f_{m+1}(x))}{\left(\frac{\partial P}{\partial z_{m+1}}(f_1(x), \dots, f_{m+1}(x)) \right)^2}$$

is a holomorphic function in E .

For every $\varepsilon > 0$ we can compute κ such that if the coefficients of the Taylor expansion of $P(f_1, \dots, f_{m+1})$ at zero up to order κ vanish, then $\|C\|_E < \frac{\varepsilon}{2}$. Indeed, it is sufficient to compute κ such that if the coefficients of the Taylor expansion of $P(f_1, \dots, f_{m+1})$ at zero up to order κ vanish, then $\|P(f_1, \dots, f_{m+1})\|_E < \frac{\varepsilon \tilde{c}^2}{2}$. This can be done in the same way as computing $N(\theta, \varepsilon, \tilde{R})$ in the last but one paragraph of Section 3.2.1.

If $\|C\|_E < \frac{\varepsilon}{2}$, then there is $\gamma > 0$ such that for Taylor polynomials f_1^ν, \dots, f_m^ν of f_1, \dots, f_m , respectively, with $\|f_i - f_i^\nu\|_E < \gamma$, $i = 1, \dots, m$, we have

$$\left\| \frac{P(f_1^\nu, \dots, f_m^\nu, f_{m+1})}{\left(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, f_{m+1}) \right)^2} \right\|_E < \varepsilon.$$

The number γ is effectively computable (by means of bounds for the partial derivatives of P and $\frac{\partial P}{\partial z_{m+1}}$ on the polydisc in \mathbf{C}^{m+1} centered at the origin, of radius $\tilde{K} + 1$, where \tilde{K} is a common bound for f_1, \dots, f_{m+1} on E).

Therefore, by Lemma 3.4 and Remark 3.5, for every $\delta > 0$ we can compute $\varepsilon > 0$ such that if $\|C\|_E < \frac{\varepsilon}{2}$, then there are polynomials f_1^ν, \dots, f_m^ν and a Nash function f_{m+1}^ν such that $\|f_i - f_i^\nu\|_E < \delta$, for $i = 1, \dots, m+1$, and $P(f_1^\nu, \dots, f_{m+1}^\nu) = 0$. Hence, for every $\delta > 0$ we can compute κ such that if only the coefficients of the Taylor expansion of $P(f_1, \dots, f_{m+1})$ up to order κ vanish, then there are polynomials f_1^ν, \dots, f_m^ν and a Nash function f_{m+1}^ν whose distance from f_1, \dots, f_{m+1} , respectively, is bounded by δ and $P(f_1^\nu, \dots, f_{m+1}^\nu) = 0$.

Recall there are effectively computable $Q_2, \dots, Q_s \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ such that

$$V \setminus R^{-1}(0) = \{R \neq 0, P = 0, z_{m+j}R = Q_j, j = 2, \dots, s\}$$

(cf. [19] p. 233, [16], [21]).

Now compute nonnegative $\delta < \frac{1}{\nu}$ so small that if $\|f_i - f_i^\nu\|_E < \delta$, for $i = 1, \dots, m+1$, then $R(f_1^\nu, \dots, f_m^\nu)$ does not vanish in E and

$$\left\| \frac{Q_j(f_1, \dots, f_{m+1})}{R(f_1, \dots, f_m)} - \frac{Q_j(f_1^\nu, \dots, f_{m+1}^\nu)}{R(f_1^\nu, \dots, f_m^\nu)} \right\|_E < \frac{1}{2\nu}.$$

Next compute κ such that if only the coefficients of the Taylor expansion of $P(f_1, \dots, f_{m+1})$ up to order κ vanish, then there are polynomials f_1^ν, \dots, f_m^ν and a Nash function f_{m+1}^ν whose distance from f_1, \dots, f_{m+1} , respectively, on E is bounded by δ and $P(f_1^\nu, \dots, f_{m+1}^\nu) = 0$. Check whether all coefficients of the Taylor expansion of $P(f_1, \dots, f_{m+1})$ up to order κ vanish; if not, then return " $F(U) \not\subseteq V$ ".

Let f_1^ν, \dots, f_m^ν be polynomials and f_{m+1}^ν a Nash function whose distance from f_1, \dots, f_{m+1} , respectively, on E is bounded by δ and $P(f_1^\nu, \dots, f_{m+1}^\nu) = 0$. Then, by Lemma 3.3 with $L(z_{m+1}, \dots, z_{m+s}) = z_{m+1}$ and $Z = \mathcal{V}(V, H)$ where $H = (f_1^\nu, \dots, f_m^\nu)$, we conclude (cf. the discussion preceding Algorithm 1) that there are holomorphic functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ such that the image of $(f_1^\nu, \dots, f_{m+s}^\nu)$ is contained in V . Consequently, the functions $f_1^\nu, \dots, f_{m+s}^\nu$ satisfy the equations

$$f_{m+j}^\nu R(f_1^\nu, \dots, f_m^\nu) = Q_j(f_1^\nu, \dots, f_{m+1}^\nu), \text{ for } j = 2, \dots, s$$

(which in particular implies that $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ are not only holomorphic but Nash as well).

The functions f_1, \dots, f_{m+s} may not satisfy these equations (when $F(U) \not\subseteq V$). But if the coefficients of the Taylor expansion of

$$f_{m+j}R(f_1, \dots, f_m) - Q_j(f_1, \dots, f_{m+1})$$

at zero up to sufficiently high order β vanish, for $j = 2, \dots, s$, then $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ are close to f_{m+2}, \dots, f_{m+s} if $f_1^\nu, \dots, f_{m+1}^\nu$ are close to f_1, \dots, f_{m+1} , respectively. **More precisely, we compute β such that if the vanishing condition is satisfied up to order β , then**

$$\|f_{m+j}R(f_1, \dots, f_m) - Q_j(f_1, \dots, f_{m+1})\|_E < \frac{\tilde{c}}{2\nu}.$$

This can be done in the same way as computing $N(\theta, \varepsilon, \hat{R})$ in the last but one paragraph of Section 3.2.1. **If the vanishing condition is not satisfied, then we have $F(U) \not\subseteq V$. Otherwise, $\|f_{m+j} - f_{m+j}^\nu\|_E \leq$**

$$\left\| f_{m+j} - \frac{Q_j(f_1, \dots, f_{m+1})}{R(f_1, \dots, f_m)} \right\|_E + \left\| \frac{Q_j(f_1, \dots, f_{m+1})}{R(f_1, \dots, f_m)} - \frac{Q_j(f_1^\nu, \dots, f_{m+1}^\nu)}{R(f_1^\nu, \dots, f_m^\nu)} \right\|_E < \frac{1}{\nu},$$

for $j = 2, \dots, s$.

When we know that the approximation exists (on $U_0 = E$), the polynomials P_i^ν , can be computed as follows. For $i = 1, \dots, m$, set $P_i^\nu(x, z_i) = z_i - f_i^\nu(x)$. Next define

$$V^\nu = \{(x, z) \in \mathbf{C}_x^n \times \mathbf{C}_z^{m+s} : z \in V, z_i = f_i^\nu(x), \text{ for } i = 1, \dots, m\},$$

where $z = (z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s})$. Finally, for $i = 1, \dots, s$, take $P_{m+i}^\nu \in (\mathbf{C}[x])[z_{m+i}]$ to be the optimal polynomial describing the image of the projection of V^ν to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}$. (This projection is proper so the image is an algebraic variety.)

Let us present the main algorithm solving the problem in the general case. First, the algorithm reduces the problem, by recursive calls, to the case where V is purely dimensional. Then it either detects that $F(U) \not\subseteq V$ or reduces the problem to the case where V is a hypersurface. It is possible that after the reduction the output will turn out to be " $F(U) \not\subseteq V$ ". By reducing to the

case of a hypersurface we mean that the algorithm produces a single polynomial $P \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ monic in z_{m+1} and a new holomorphic map $F^* : U \rightarrow \mathbf{C}^{m+1}$ such that the following hold. The composition $\frac{\partial P}{\partial z_{m+1}} \circ F^*$ is a non-zero function and if $P \circ F^* \neq 0$ then $F(U) \not\subseteq V$. Moreover, from any Nash approximation $F^{*,\nu}$ of F^* satisfying $P \circ F^{*,\nu} = 0$ and $\frac{\partial P}{\partial z_{m+1}} \circ F^{*,\nu} \neq 0$ we can recover a Nash map F^ν into V such that F^ν approximates F if $F(U)$ is close to V (even if $F(U) \not\subseteq V$).

If the algorithm has produced P, F^* as described above, then, using $P, \frac{\partial P}{\partial z_{m+1}}, F^*$ it either recognizes that $F(0)$ is a regular point of V (this case has been discussed separately above) or performs an operation the outcome of which is one of the two following things. The first thing is the detection of $F(U) \not\subseteq V$, the second one is confirmation of the fact that $F(U)$ is close to V together with a Nash map $F^{*,\nu}$ approximating F^* and satisfying $P \circ F^{*,\nu} = 0$ and $\frac{\partial P}{\partial z_{m+1}} \circ F^{*,\nu} \neq 0$.

The operation consists of two stages. The first stage is checking the fulfillment of a condition which implies that $F(U)$ is close to V , and whose negation gives $F(U) \not\subseteq V$. The second stage is producing another holomorphic map g depending on smaller number of variables than F and a new algebraic set X with the following properties. If the image of g is not contained in X then $P \circ F^* \neq 0$ (and then $F(U) \not\subseteq V$). If there is a Nash approximation of g into X , then there is a Nash approximation \hat{F} of F^* into \mathbf{C}^{m+1} and a Nash function C close to zero such that $P \circ \hat{F} = C \cdot (\frac{\partial P}{\partial z_{m+1}} \circ \hat{F})^2$ (so, by Lemma 3.4, we have a Nash approximation $F^{*,\nu}$ of F^* with $P \circ F^{*,\nu} = 0$ and $\frac{\partial P}{\partial z_{m+1}} \circ F^{*,\nu} \neq 0$, and then, as mentioned above, we obtain a Nash map F^ν into V approximating F).

To obtain a Nash approximation of g into X or to find out that the image of g is not contained in X the algorithm calls itself recursively with g and X . The details are as follows.

Algorithm 2

1. If $n = 0$ or $m = 0$, then proceed as discussed above.
2. Compute the equidimensional decomposition $V_{(0)} \cup \dots \cup V_{(m)}$ of V .
3. For $j = 0, \dots, m - 1$ call the algorithm recursively with $\nu, V_{(j)}, F$ and if it returns an approximating map for at least one j , then stop. If for every $j = 0, \dots, m - 1$ the algorithm returns " $F(U) \not\subseteq V_{(j)}$ ", then go to Step 4.
4. After a generic linear change of the coordinates in $\mathbf{C}^{\hat{m}}$ we have:
 - (x) $V_{(m)} \subset \mathbf{C}^{\hat{m}} \approx \mathbf{C}^m \times \mathbf{C}^s$ has proper projection onto \mathbf{C}^m ,
 - (y) generic fibers in $V_{(m)}$ and in $\Phi(V_{(m)})$ over \mathbf{C}^m have the same cardinalities, where $\Phi(z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s}) = (z_1, \dots, z_m, z_{m+1})$.
 Apply such a change of the coordinates.
5. Calculate the optimal polynomial $P \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ describing $\Phi(V_{(m)}) \subset \mathbf{C}^m \times \mathbf{C}$. Calculate the discriminant $R \in \mathbf{C}[z_1, \dots, z_m]$ of P .
6. Call the algorithm recursively with $\nu, V_{(m)} \cap \{R = 0\}, F$. If it returns an approximation, then stop. Otherwise one has $F(U) \not\subseteq V_{(m)} \cap \{R = 0\}$, and we **decide whether** $F(U) \not\subseteq V_{(m)}$ (return " $F(U) \not\subseteq V$ ") or $R(f_1, \dots, f_m) \neq 0$

(cf. Remark 3.6). If $R(f_1, \dots, f_m) \neq 0$ then $P(f_1, \dots, f_m, f_{m+1}) \neq 0$ (return " $F(U) \not\subseteq V$ ") or $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1}) \neq 0$ (go to Step 7).

7. If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(0) \neq 0$, then either $F(0) \notin V_{(m)}$ (return " $F(U) \not\subseteq V$ ") or $F(0) \in \text{Reg}(V_{(m)})$ (case discussed above). If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(0) = 0$, then apply a linear change of the coordinates in \mathbf{C}^n after which the following holds: $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(x) = \hat{H}(x)W(x)$ in some neighborhood of $0 \in \mathbf{C}^n$, where \hat{H} is a holomorphic function, $\hat{H}(0) \neq 0$ and W is a monic polynomial in x_n with holomorphic coefficients vanishing at $0 \in \mathbf{C}^{n-1}$, depending on $x' = (x_1, \dots, x_{n-1})$. Put $d = \deg(W)$.

8. Choose polydiscs $E_1 \subset \mathbf{C}^{n-1}$, $E_2 \subset \mathbf{C}$, centered at zero, and a real number $\tilde{c} > 0$ such that $E_1 \times E_2$ is contained in the open neighborhood of the origin in which Step 7 has been performed, $\inf_{E_1 \times E_2} |\hat{H}| > 0$, $\inf_{E_1 \times \partial E_2} |R(f_1, \dots, f_m)| > \tilde{c}$, and $\inf_{E_1 \times \partial E_2} |\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})| > \tilde{c}$.

9. Compute $Q_2, \dots, Q_s \in (\mathbf{C}[z_1, \dots, z_m])[z_{m+1}]$ such that

$$V_{(m)} \setminus R^{-1}(0) = \{R \neq 0, P = 0, z_{m+j}R = Q_j, j = 2, \dots, s\}.$$

Compute $\kappa \in \mathbf{N}$ such that:

(u) if all the coefficients of the Taylor expansion of $P(f_1, \dots, f_{m+1})$ at zero up to order κ vanish, then $\|P(f_1, \dots, f_{m+1})\|_{E_1 \times E_2} < \theta(\nu)$,

(v) if all the coefficients of the Taylor expansion of

$$f_{m+j}R(f_1, \dots, f_m) - Q_j(f_1, \dots, f_{m+1})$$

at zero up to order κ vanish, then $\|f_{m+j}R(f_1, \dots, f_m) - Q_j(f_1, \dots, f_{m+1})\|_{E_1 \times E_2} < \eta(\nu)$, for $j = 2, \dots, s$.

(How to choose $\theta(\nu), \eta(\nu)$ is described below, where accuracy of approximation is discussed; κ can be computed similarly to computing $N(\theta, \varepsilon, \tilde{R})$ in the last but one paragraph of Section 3.2.1.) Check whether all coefficients of the expansions vanish. If not, then return " $F(U) \not\subseteq V$ ", otherwise go to Step 10.

10. Divide f_i by $(W)^2$ to obtain $f_i = (W)^2 H_i + r_i$ in some neighborhood of $E_1 \times E_2 \subset \mathbf{C}^n$, $i = 1, \dots, m+1$. Here H_i are holomorphic functions and r_i are polynomials in x_n with holomorphic coefficients depending on x' such that $\deg(r_i) < 2d$.

11. Treating H_i , $i = 1, \dots, m+1$, and all the coefficients of W, r_1, \dots, r_{m+1} as new variables (except for the leading coefficient 1 of W) apply the division procedure for polynomials to obtain:

$$\begin{aligned} P(W^2 H_1 + r_1, \dots, W^2 H_{m+1} + r_{m+1}) &= \tilde{W}W^2 + x_n^{2d-1}T_1 + x_n^{2d-2}T_2 + \dots + T_{2d}, \\ \frac{\partial P}{\partial z_{m+1}}(W^2 H_1 + r_1, \dots, W^2 H_{m+1} + r_{m+1}) &= \tilde{S}W + x_n^{d-1}T_{2d+1} + x_n^{d-2}T_{2d+2} + \dots + T_{3d}. \end{aligned}$$

Here T_1, \dots, T_{3d} are polynomials depending only on the variables standing for the coefficients of W, r_1, \dots, r_{m+1} . Let g be the mapping (in $n-1$ variables) whose components are all these coefficients.

12. Call the algorithm recursively with ν, V, F replaced by $\mu(\nu), \{T_1 = \dots = T_{3d} = 0\}, g$, respectively. (How to choose $\mu(\nu)$ is described below.) If the algorithm detects that the image of g is not contained in $\{T_1 = \dots = T_{3d} = 0\}$, then $P(f_1, \dots, f_{m+1}) \neq 0$ (return " $F(U) \not\subseteq V$ "). Otherwise, for every component $c(x')$ of $g(x')$ one obtains a monic polynomial $Q_c^\mu(x', t_c) \in (\mathbf{C}[x'])[t_c]$ which put

in place of $P_i^\nu(x, z_i)$ satisfies (a) and (b) above with $\nu, x, z_i, f_i^\nu, F, F^\nu$ replaced by $\mu, x', t_c, c^\mu, g, g^\mu$, respectively. Here, for every c, c^μ is a Nash function approximating c in $\overline{E'_1} \subset E_1$ (where E'_1 is an open polydisc centered at the origin and independent of μ) such that the mapping g^μ obtained by replacing every c of g by c^μ satisfies $T_1 \circ g^\mu = \dots = T_{3d} \circ g^\mu = 0$.

13. Approximate H_i for $i = 1, \dots, m$, by polynomials H_i^μ . (Accuracy of this approximation is discussed below.) Let $W_\mu, r_{1,\mu}, \dots, r_{m+1,\mu}$ be the polynomials in x_n defined by replacing the coefficients of W, r_1, \dots, r_{m+1} by their Nash approximations (i.e. the components of g^μ) determined in Step 12. Using Q_c^μ (for all c) and H_i^μ one can calculate $P_i^\nu \in (\mathbf{C}[x])[z_i]$, for $i = 1, \dots, m$, satisfying (b) and (a) with $f_i^\nu = H_i^\mu(W_\mu)^2 + r_{i,\mu}$ being the i 'th component of the mapping F^ν (whose last $\hat{m} - m$ components are determined by $P_{m+1}^\nu, \dots, P_{\hat{m}}^\nu$ obtained in the next step). To calculate P_i^ν , for $i = 1, \dots, m$, perform the ring operations in the construction of f_i^ν in their implicit representation (see Lemma 2.1).

14. Put $V^\nu = \{(x, z) \in \mathbf{C}_x^n \times \mathbf{C}_z^{m+s} : z \in V_{(m)}, P_i^\nu(x, z_i) = 0 \text{ for } i = 1, \dots, m\}$, where $z = (z_1, \dots, z_m, z_{m+1}, \dots, z_{m+s})$. For $i = 1, \dots, s$, take $P_{m+i}^\nu \in (\mathbf{C}[x])[z_{m+i}]$ to be the optimal polynomial describing the image of the projection of V^ν to $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+i}}$.

Let us list the main differences between Algorithm 1 and Algorithm 2. First we need Steps 1, 2, 3 in Algorithm 2 because without zero-test we cannot check which equidimensional components of V (if any) contain the image of F . In Algorithm 1 we pick such a component in Step 1.

Next, Algorithm 2 calls itself recursively in Step 6 because without zero-test one cannot check whether $R(f_1, \dots, f_m) \neq 0$. Observe that if $F(U) \subset V_{(m)}$, then detecting that $R(f_1, \dots, f_m) \neq 0$ (see Step 4 of Algorithm 1) is equivalent to detecting that $F(U) \not\subseteq V_{(m)} \cap \{R = 0\}$ (see Step 6 of Algorithm 2).

Moreover, in Step 9 of Algorithm 2 we compute $\kappa, \theta(\nu), \eta(\nu)$ as there is still the possibility that $f_{m+j}R(f_1, \dots, f_m) - Q_j(f_1, \dots, f_{m+1}) \neq 0$, for some j , or $P(f_1, \dots, f_m, f_{m+1}) \neq 0$ which implies $F(U) \not\subseteq V$. This does not appear in Algorithm 1, where we know that $F(U) \subset V$.

3.2.3.1 Controlling accuracy of approximation

Let us explain how to control accuracy of approximation of F by F^ν on $U_0 = E'_1 \times E_2$. In Step 7, we assume that $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})(0) = 0$, since in the other case we have either $F(U) \not\subseteq V$ or $F(0) \in \text{Reg}(V_{(m)})$ which has already been discussed. We will show how to choose $\theta(\nu), \eta(\nu)$ in Step 9 and how to chose $\mu(\nu)$ and the bounds for the distance between H_i and H_i^μ in Steps 12, 13, respectively, to ensure the required precision of approximation.

First compute $\delta < \frac{1}{\nu}$ and $\eta(\nu)$ such that if the inequality in (v), Step 9, holds and if for some polydisc $G \subset E_1$ centered at zero we have $\|f_i - f_i^\nu\|_{G \times E_2} < \delta$, $i = 1, \dots, m+1$, then for $f_{m+j}^\nu, j = 2, \dots, s$, satisfying

$$f_{m+j}^\nu R(f_1^\nu, \dots, f_m^\nu) = Q_j(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu)$$

on (some neighborhood of) $\overline{G \times E_2}$ we have $\|f_{m+j} - f_{m+j}^\nu\|_{\overline{G \times E_2}} < \frac{1}{\nu}$. To do this, note that if the inequality in (v) holds, then $\|f_{m+j} - \frac{Q_j(f_1, \dots, f_m, f_{m+1})}{R(f_1, \dots, f_m)}\|_{E_1 \times \delta E_2} <$

$\frac{\eta(\nu)}{\tilde{c}}$, so for f_{m+j}^ν as above we have $\|f_{m+j} - f_{m+j}^\nu\|_{G \times E_2} \leq \frac{\eta(\nu)}{\tilde{c}} + \left\| \frac{Q_j(f_1, \dots, f_m, f_{m+1})}{R(f_1, \dots, f_m)} - \frac{Q_j(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu)}{R(f_1^\nu, \dots, f_m^\nu)} \right\|_{G \times \partial E_2}$. Now it is sufficient to take $\eta(\nu) < \frac{\tilde{c}}{2\nu}$ and δ so small that the second term of the righthand side of the last inequality is bounded by $\frac{1}{2\nu}$. Here recall that $\inf_{E_1 \times \partial E_2} |R(f_1, \dots, f_m)| > \tilde{c}$; in particular, we choose δ so small that $\inf_{\bar{G} \times \partial E_2} |R(f_1^\nu, \dots, f_m^\nu)| > \frac{\tilde{c}}{2}$.

Next, compute ε in Lemma 3.4 (cf. Remark 3.5) with d equal to the degree of P in z_{m+1} and M such that $\frac{M}{2}$ is a common bound for f_{m+1} and the coefficients of $P(f_1, \dots, f_m, z_{m+1}) \in \mathcal{O}(E_1 \times E_2)[z_{m+1}]$. Define $\theta(\nu) := \min\{\frac{\varepsilon\tilde{c}^2}{8}, \frac{\delta\tilde{c}^2}{32N}\}$, where $\frac{N}{2}$ is a bound for $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})$. We will show that with $\eta(\nu), \theta(\nu)$ picked above and $\frac{1}{\mu(\nu)}, \|H_i^\mu - H_i\|_{E_1 \times E_2}$ small enough we have $\|f_i - f_i^\nu\|_{E_1 \times E_2} < \delta$, for $i = 1, \dots, m+1$.

From $f_i = H_i(W)^2 + r_i$ and $f_i^\nu = H_i^\mu(W_\mu)^2 + r_{i,\mu}$ (cf. Steps 10, 13), for $i = 1, \dots, m$, we can clearly determine how close H_i^μ, g^μ to H_i, g have to be to ensure that $\|f_i - f_i^\nu\|_{E_1 \times E_2} < \delta$, for $i = 1, \dots, m$. That is, we have upper bounds for $\frac{1}{\mu(\nu)}$ and $\|H_i^\mu - H_i\|_{E_1 \times E_2}$ which imply $\|f_i - f_i^\nu\|_{E_1 \times E_2} < \delta$, for $i = 1, \dots, m$. Let us show that (for possibly smaller bounds) we also have $\|f_{m+1} - f_{m+1}^\nu\|_{E_1 \times E_2} < \delta$.

To do this, denote $\bar{f}^\nu = H_{m+1}^\mu W_\mu^2 + r_{m+1,\mu}$, where H_{m+1}^μ is a polynomial approximation of H_{m+1} . We can compute an upper bound for $\frac{1}{\mu(\nu)}$ and $\|H_i^\mu - H_i\|_{E_1 \times E_2}$ which guarantee that

$$P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = C^\nu \left(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) \right)^2,$$

on $E_1' \times E_2$, where C^ν is a holomorphic function. Indeed, observe that $W(0, x_n) = x_n^d$ and $W_\mu(0, x_n)$ has the form $W_\mu(0, x_n) = x_n^d + w_\mu(x_n)$. Compute how close g^μ, H_i^μ to g, H_i have to be to ensure that $W_\mu(0, \cdot)$ has $d = \deg(W)$ roots in E_2 (counted with multiplicities; by the Rouché Theorem it is sufficient to ensure that $|x_n|^d > |w_\mu(x_n)|$ on ∂E_2) and

$$(3.5) \quad \left\| \frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) - \frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1}) \right\|_{E_1' \times E_2} < \frac{\tilde{c}}{2}.$$

If the last two conditions hold, then, by the fact that

$$\inf_{E_1' \times \partial E_2} \left| \frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1}) \right| > \tilde{c}$$

and by the Rouché Theorem, $\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)(x', \cdot)$ has d roots (counted with multiplicities) in E_2 for every $x' \in E_1'$ (and no root in ∂E_2). But recall that $\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)$ is divisible by W_μ (because $T_j \circ g^\mu = 0$, for $j = 2d+1, \dots, 3d$, cf. Steps 11, 12, 13), and $W_\mu(0, \cdot)$ has d roots. Hence, for every $x' \in E_1'$ we have that $\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)(x', \cdot)$ and $W_\mu(x', \cdot)$ have the same roots with the same multiplicities. On the other hand, $P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)$ is divisible by $(W_\mu)^2$ (because $T_j \circ g^\mu = 0$, for $j = 1, \dots, 2d$, cf. Steps 11, 12, 13)

so

$$P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) = C^\nu \left(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu) \right)^2,$$

on $E'_1 \times E_2$, where C^ν is a holomorphic function, as claimed above.

Now, improving the obtained upper bounds for $\frac{1}{\mu(\nu)}$ and $\|H_i - H_i^\mu\|_{E'_1 \times E_2}$, which can be done effectively, we have that

$\|\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)\|_{E'_1 \times E_2} < N$, $\|P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)\|_{E'_1 \times E_2} < 2\theta(\nu)$ (recall (u) in Step 9) and $\|f_{m+1} - \bar{f}^\nu\|_{E'_1 \times E_2} < \frac{\delta}{2}$, and $\|\bar{f}^\nu\|_{E'_1 \times E_2} < M$, and the coefficients of $P(f_1^\nu, \dots, f_m^\nu, z_{m+1}) \in \mathcal{O}(E'_1 \times E_2)[z_{m+1}]$ are bounded by M . (Recall that $\frac{N}{2}$ is a bound for $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})$ and $\frac{M}{2}$ is a common bound for f_{m+1} and the coefficients of $P(f_1, \dots, f_m, z_{m+1}) \in \mathcal{O}(E_1 \times E_2)[z_{m+1}]$.)

When this is done then, by (3.5) and by $\inf_{E'_1 \times \partial E_2} |\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_{m+1})| > \tilde{c}$, we have

$$\|C^\nu\|_{E'_1 \times E_2} = \|C^\nu\|_{E'_1 \times \partial E_2} \leq \frac{\|P(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)\|_{E'_1 \times E_2}}{\inf_{E'_1 \times \partial E_2} |(\frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu))^2|} < \frac{8\theta(\nu)}{\tilde{c}^2}.$$

Consequently, (cf. the definition of $\theta(\nu)$) we obtain $\|C^\nu\|_{E'_1 \times E_2} < \varepsilon$ and

$$2\|C^\nu \cdot \frac{\partial P}{\partial z_{m+1}}(f_1^\nu, \dots, f_m^\nu, \bar{f}^\nu)\|_{E'_1 \times E_2} \leq \frac{\delta}{2}.$$

Hence, by Lemma 3.4, there is $f_{m+1}^\nu \in \mathcal{O}(E'_1 \times E_2)$ with $P(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu) = 0$ (so f_{m+1}^ν is Nash as its graph is contained in a Nash hypersurface) and $\|f_{m+1}^\nu - \bar{f}^\nu\|_{E'_1 \times E_2} \leq \frac{\delta}{2}$ therefore $\|f_{m+1}^\nu - f_{m+1}\|_{E'_1 \times E_2} < \delta$. Note that $\{(x, z_{m+1}) : P(f_1^\nu(x), \dots, f_m^\nu(x), z_{m+1}) = 0\}$ is contained in the image of the projection of V^ν defined in Step 14 onto $\mathbf{C}_x^n \times \mathbf{C}_{z_{m+1}}$. Thus we can effectively bound $\frac{1}{\mu(\nu)}$ and $\|H_i - H_i^\mu\|_{E'_1 \times E_2}$ from above to ensure that $\|f_{m+1} - f_{m+1}^\nu\|_{E'_1 \times E_2} < \delta$.

With $f_1^\nu, \dots, f_{m+1}^\nu$ as above, we know that there are holomorphic functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ such that the image of $(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu, \dots, f_{m+s}^\nu)$ is contained in $V(m)$. (This follows, in view of $\inf_{E'_1 \times \partial E_2} |R(f_1^\nu, \dots, f_m^\nu)| > \frac{\tilde{c}}{2}$, by Lemma 3.3 with $L(z_{m+1}, \dots, z_{m+s}) = z_{m+1}$ and $Z = \mathcal{V}(V(m), H)$ where $H = (f_1^\nu, \dots, f_m^\nu)$; cf. also the discussion preceding Algorithm 1). Therefore we have

$$f_{m+j}^\nu R(f_1^\nu, \dots, f_m^\nu) = Q_j(f_1^\nu, \dots, f_m^\nu, f_{m+1}^\nu)$$

on some neighborhood of $\overline{E'_1 \times E_2}$, for $j = 2, \dots, s$ (cf. Step 9; in particular the functions $f_{m+2}^\nu, \dots, f_{m+s}^\nu$ are not only holomorphic but Nash as well). As we have proved above, this implies that $\|f_{m+j}^\nu - f_{m+j}\|_{E'_1 \times E_2} < \frac{1}{\nu}$, for $j = 2, \dots, s$, which completes the discussion on error control.

3.2.3.2 Bound for the degree of output polynomials

Finally, let us prove the claim: there is a bound, independent of ν , for the degree of P_i^ν in z_i . We may assume that $F(U) \subseteq V$ as otherwise there is ν_0 such that for every $\nu > \nu_0$ the algorithm returns " $F(U) \not\subseteq V$ ", and there is nothing to

prove. Consider the linear ordering of \mathbf{N}^2 defined by $(n, m) < (n', m')$ iff $n < n'$ or $(n = n' \text{ and } m < m')$. Fix $(n, m) \in \mathbf{N}^2$. Let V be an algebraic subvariety of \mathbf{C}^m with $\dim(V) = m$ and let $F : U \rightarrow V$ be any holomorphic map, where U is an open neighborhood of 0 in \mathbf{C}^n . If $n = 0$ or $m = 0$, then there is nothing to prove because F is a constant map. Assume that the claim is true for every holomorphic map $\tilde{F} : \tilde{U} \rightarrow \tilde{V}$, where \tilde{U} is an open neighborhood of 0 in \mathbf{C}^l and $\dim(\tilde{V}) = k$ and $(l, k) < (n, m)$. We will show that the claim is also true for F , which, by induction, will complete the proof.

We may assume that $F(U) \subseteq V_{(m)}$ because if $F(U)$ is contained in the lower-dimensional part of V , then the proof is complete by the induction hypothesis. Furthermore, in Step 6, we may assume that $F(U) \not\subseteq \{R = 0\}$ because otherwise $F(U)$ is contained in some variety of dimension smaller than m and again the induction hypothesis completes the proof. This implies that $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1}) \neq 0$ and either $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})(0) \neq 0$ or $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})(0) = 0$.

If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})(0) \neq 0$, then $F(0) \in \text{Reg}(V_{(m)})$ and the polynomial P_i^ν is linear in z_i for $i = 1, \dots, m$. (Recall that in the case $F(0) \in \text{Reg}(V_{(m)})$ the algorithm has been discussed separately above.) For $i = m+1, \dots, m+s$, the number $\deg P_i^\nu$ is bounded by the cardinality of the generic fiber in V over \mathbf{C}^m . Indeed, by construction of P_i^ν for $i = m+1, \dots, m+s$, $\deg P_i^\nu$ does not exceed the number of points in the generic fiber in V^ν over x which (by linearity of P_i^ν in z_i for $i = 1, \dots, m$) is bounded by the cardinality of the generic fiber in V over \mathbf{C}^m .

If $\frac{\partial P}{\partial z_{m+1}}(f_1, \dots, f_m, f_{m+1})(0) = 0$, then we can repeat the argument which was given for Algorithm 1 to prove the existence of bounds independent of ν for $\deg P_i^\nu$.

Remark 3.9 Suppose that R computed in Step 5 is reducible, i.e. $R = R_1^{\alpha_1} \cdot \dots \cdot R_k^{\alpha_k}$, where R_1, \dots, R_k are some relatively prime non-constant polynomials. Observe that instead of calling the algorithm with $V_{(m)} \cap \{R = 0\}$ in Step 6, we may call it simultaneously (i.e. in parallel) with every $V_{(m)} \cap \{R_j = 0\}$ (each of which is simpler than the original $V_{(m)} \cap \{R = 0\}$) and recover the output for $V_{(m)} \cap \{R = 0\}$ from the output for $V_{(m)} \cap \{R_j = 0\}$, for $j = 1, \dots, k$. To do this, we need to compute non-trivial factors of R . Let us discuss this problem when $V_{(m)}$ is of the form $V_{(m)} = \tilde{V}_{(m+1)} \cap \{\tilde{R} = 0\}$ where $\tilde{V}_{(m+1)} \subset \mathbf{C}^{m+1} \times \mathbf{C}^{s-1}$ has pure dimension $m+1$ and has proper projection onto \mathbf{C}^{m+1} and \tilde{R} is the discriminant of the optimal polynomial for $\tilde{\Phi}(\tilde{V}_{(m+1)})$, and $\tilde{\Phi}(z_1, \dots, z_{m+1}, z_{m+2}, \dots, z_{m+s}) = (z_1, \dots, z_{m+1}, z_{m+2})$; we deal with such $V_{(m)}$ after calling Algorithm 2 recursively with $\tilde{V}_{(m+1)}$.

We may assume (changing the coordinates in $\mathbf{C}^{m+1} \approx \mathbf{C}^m \times \mathbf{C}$) that $\{\tilde{R} = 0\} \subset \mathbf{C}^m \times \mathbf{C}$ has proper projection onto \mathbf{C}^m . Set $V_{\tilde{R}} = \{(z_1, \dots, z_m, z_{m+1}) \in \mathbf{C}^m \times \mathbf{C} : \tilde{R}(z_1, \dots, z_m, z_{m+1}) = 0\}$. Choose $R^* \in \mathbf{C}[z_1, \dots, z_m]$ such that $\{R^* = 0\} \subset \mathbf{C}^m$ is the set of points over which the fiber in $V_{\tilde{R}}$ has not the maximal cardinality. If the discriminant of \tilde{R} is non-zero, then define R^* to be

the discriminant of \tilde{R} .

It is clear that (after a generic change of the coordinates in $\mathbf{C}_{z_{m+1}, \dots, z_{m+s}}^s$), $V_{(m)} = \tilde{V}_{(m+1)} \cap \{\tilde{R} = 0\}$ satisfies (x), (y) of Step 4 and that $\{R^* = 0\} \subset \{R = 0\}$. Therefore (reduced) factors of R^* are also factors of R . But R^* is an iterated discriminant and there are direct methods for computing factors of iterated discriminants (cf. [22]) which in many cases are much faster than computing and then factorizing R^* .

3.2.4 Example

The aim of this subsection is to present an example illustrating Algorithm 1. We have chosen input data in such a way that computations are short enough to be presented here and still not quite trivial.

Let

$$P(z_1, z_2, z_3, z_4, z_5, z_6) = z_6^5 + z_6 z_5 + z_5^6 + z_1^2 + z_2^2 + z_3^2 + z_4^2,$$

$$V = \{(z_1, z_2, z_3, z_4, z_5, z_6) \in \mathbf{C}^6 : P(z_1, z_2, z_3, z_4, z_5, z_6) = 0\}.$$

Let

$$m_1(x_1, x_2) = \frac{1}{2}x_2 - \frac{1}{4}x_1 \sin\left(\frac{1}{4}x_1\right), \quad m_2(x_1, x_2) = \frac{1}{2}x_2 - \frac{1}{4}x_1 \cos\left(\frac{1}{4}x_1\right),$$

$$p = 1 + (1 - m_1^{19})^{\frac{1}{6}}.$$

Define

$$F(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2), f_4(x_1, x_2), f_5(x_1, x_2), f_6(x_1, x_2))$$

by setting

$$f_6 = m_1^5, \quad f_5 = m_1(1 - m_1^{19})^{\frac{1}{6}}, \quad f_1 = ip^{\frac{1}{2}}m_2^3, \quad f_2 = ip^{\frac{1}{2}}(m_1^3 + m_2^3),$$

$$f_3 = -\frac{i}{2}(m_1^3 + m_2^3) + im_2^3p, \quad f_4 = \frac{1}{2}(m_1^3 + m_2^3) + m_2^3p.$$

It is easy to verify that the image of F is contained in V (but not in $\text{Sing}(V)$). We will illustrate Algorithm 1 by computing a Nash approximation of F . Given the formulas above we know much more about F than just the input data described in Section 3.2.1. This extra information could be used to simplify the computations, but our aim is to show how to compute approximations when we have nothing but $P, \text{Expand}_{f_j}, M_{f_j}, U_{f_j}$. Therefore, when performing the algorithm, we act as if the formulas defining the components of F could be used only for computing the coefficients of the Taylor expansion of these components (i.e. the output of the procedure *Expand*). The reader might find the formulas useful to confirm (with help of a pocket calculator) the correctness of numerical constants such as upper and lower bounds of functions appearing in the sequel. Then the steps of the algorithm can be performed by hand without using computer programs. Note that in practice explicit formulas might not exist, and when exist, they might not be easy to guess.

We assume $U_{f_j} = D_{\frac{3}{2}} \times D_{\frac{3}{2}}$, for every j , where $D_r \subset \mathbf{C}$ is the disc of radius r centered at 0. On $D_{\frac{3}{2}}$ we have $|\sin(\frac{1}{4}x_1)| < 0.4$, so $|m_1(x_1, x_2)| < 0.9$ and $|p(x_1, x_2)| < 2.1$. Also $|m_2(x_1, x_2)| < \frac{9}{8}$, hence $M_{f_6} = 0.6$, $M_{f_5} = 1$, $M_{f_1} = 2.5$, $M_{f_2} = 3.5$, $M_{f_3} = 4.5$, $M_{f_4} = 4.5$ are bounds for the components of F .

Let P, V be fixed as above (then, in particular, $\frac{\partial P}{\partial z_6}$ depends only on z_5, z_6). Note that for any holomorphic map $F(x_1, x_2) = (f_1(x_1, x_2), \dots, f_6(x_1, x_2))$ into V with $\frac{\partial P}{\partial z_6}(f_5(x_1, x_2), f_6(x_1, x_2)) = \hat{H}(x_1, x_2)(x_2 - A(x_1))$ (for some A, \hat{H} holomorphic in a neighborhood of zero, $\hat{H}(0, 0) \neq 0$) and such that at least one of f_1, \dots, f_4 does not vanish identically on the graph of A , the computations would be similar to those presented below. Hence, in fact, we discuss here the whole class of examples.

We will slightly relax our requirements regarding the output. Namely, to simplify the computations we will let some of the output polynomials $P_i^\nu(x_1, x_2, z_i)$ (cf. (b) of the output of Algorithm 1) not be monic in z_i (but their leading coefficients will be non-vanishing at the origin and their degrees in z_i will be independent of the accuracy of approximation). Our aim is to approximate F on $D_{\frac{1}{3}} \times D_{\frac{1}{3}}$ with precision 10^{-5} (i.e. $\nu = 10^5$).

Step 1: there is nothing to do because V is a hypersurface (i.e. $m = 5$).

Step 2: no change of the coordinates is necessary as P is monic in z_6 , so V has proper projection onto $\mathbf{C}_{z_1 \dots z_5}^5$. Here $\Phi = \text{id}_{\mathbf{C}^6}$. (Projecting along \mathbf{C}_{z_j} , where j is one of 1, 2, 3, 4, instead of projecting along \mathbf{C}_{z_6} leads to a large system of equations in Step 7. This is because all partial derivatives of $\frac{\partial P}{\partial z_j}(F(x_1, x_2)) = 2f_j(x_1, x_2)$, for $j = 1, \dots, 4$, up to order at least 2 vanish at 0, so the monic polynomial W obtained in Step 5 would be of degree at least 3.)

Step 3: here P is already optimal and $R = 5^5(z_5^6 + z_1^2 + z_2^2 + z_3^2 + z_4^2)^4 + 4^4 z_5^5$.

Step 4': confirm that the partial derivative of $\frac{\partial P}{\partial z_6}(f_5(x_1, x_2), f_6(x_1, x_2))$ (of order 1) with respect to x_2 at 0 is non-zero.

Step 5: since the partial derivative of $\frac{\partial P}{\partial z_6}(f_5(x_1, x_2), f_6(x_1, x_2))$ with respect to x_2 at 0 is non-zero, we have $d = 1$ and (by the Weierstrass Preparation Theorem)

$$\frac{\partial P}{\partial z_6}(f_5(x_1, x_2), f_6(x_1, x_2)) = \hat{H}(x_1, x_2)(x_2 - A(x_1)),$$

for \hat{H}, A holomorphic in some neighborhood of 0, $\hat{H}(0, 0) \neq 0$. Let us estimate the size of the domain of \hat{H}, A . Set $\tilde{h}(x_1, x_2) = \frac{\partial P}{\partial z_6}(f_5(x_1, x_2), f_6(x_1, x_2))$ and $\tilde{g}(x_2) = \frac{\tilde{h}(0, x_2)}{x_2}$. Having $\text{Expand}_{f_j}, U_{f_j}, M_{f_j}$, for $j = 5, 6$, we also have $\text{Expand}_{\tilde{h}}, U_{\tilde{h}} = D_{\frac{3}{2}} \times D_{\frac{3}{2}}$, $M_{\tilde{h}} = 1.65$ and $\text{Expand}_{\tilde{g}}, U_{\tilde{g}} = D_{\frac{3}{2}}$, $M_{\tilde{g}} = 1.2$. With these data we confirm that $\inf_{D_1 \times \partial D_1} |\tilde{h}| > 0.4 > 0$ and $\inf_{D_1} |\tilde{g}| > 0.4 > 0$. Therefore, the number of the solutions $x_2 \in D_1$ of $\tilde{h}(x_1, x_2) = 0$ is constant (counted with multiplicities) when x_1 varies in D_1 . Due to the latter inequality, $x_2 = 0$ is the only solution of $\tilde{h}(0, x_2) = 0$ in D_1 and it has multiplicity 1 because $\frac{\partial \tilde{h}}{\partial x_2}(0, 0) \neq 0$. Hence, $\{\tilde{h} = 0\} \cap (D_1 \times D_1)$ is the graph of a holomorphic function defined on D_1 . Thus, we can set $U_A = D_1, U_{\hat{H}} = D_1 \times D_1$.

Let us compute M_A . Using $Expand_{\tilde{h}}, M_{\tilde{h}}$ we check that $\inf_{D_1 \times \partial D_{\frac{1}{3}}} |\tilde{h}| > 0.05 > 0$. This implies that $\text{graph}(A) \subset D_1 \times D_{\frac{1}{3}}$, hence, we can define $M_A = \frac{1}{3}$.

Step 6: by the Weierstrass Division Theorem (cf. Section 3.2.1) we have $f_i(x_1, x_2) = (x_2 - A(x_1))^2 H_i(x_1, x_2) + a_i(x_1)x_2 + b_i(x_1)$, where H_i, a_i, b_i are holomorphic functions on $D_1 \times D_1, D_1, D_1$, respectively, for $i = 1, \dots, 6$.

Let us compute M_{a_i}, M_{b_i} . Set $r_i(x_1, x_2) = a_i(x_1)x_2 + b_i(x_1)$. We have (cf. Section 3.2.1)

$$\begin{aligned} r_i(x_1, x_2) &= \frac{1}{2\pi i} \int_{\partial D_1} \frac{f_i(x_1, s)}{(s - A(x_1))^2} \frac{(s - A(x_1))^2 - (x_2 - A(x_1))^2}{s - x_2} ds = \\ &= \frac{1}{2\pi i} \int_{\partial D_1} \frac{f_i(x_1, s)}{(s - A(x_1))^2} (s + x_2 - 2A) ds. \end{aligned}$$

Thus,

$$\begin{aligned} \|r_i(x_1, 0)\|_{D_1} &= \|b_i\|_{D_1} \leq \frac{15}{4} \|f_i\|_{D_1 \times D_1} =: M_{b_i}, \\ \|r_i(x_1, 1)\|_{D_1} &= \|a_i + b_i\|_{D_1} \leq 6 \|f_i\|_{D_1 \times D_1}, \end{aligned}$$

so

$$\|a_i\|_{D_1} \leq \frac{39}{4} \|f_i\|_{D_1 \times D_1} =: M_{a_i}.$$

Using $Expand_{f_i}, M_{f_i}$ (which bounds f_i on $D_{\frac{3}{2}} \times D_{\frac{3}{2}}$) we compute: $\|f_6\|_{D_1 \times D_1} \leq 0.2$, $\|f_5\|_{D_1 \times D_1} \leq 0.7$, $\|f_1\|_{D_1 \times D_1} \leq 0.7$, $\|f_2\|_{D_1 \times D_1} \leq 1$, $\|f_3\|_{D_1 \times D_1} \leq 1.5$, $\|f_4\|_{D_1 \times D_1} \leq 1.5$.

Step 7: we obtain

$$T_1 = Q_1 + 2Aa_1^2 + 2a_1b_1,$$

where

$Q_1 = 6a_5(a_5A + b_5)^5 + 5a_6(a_6A + b_6)^4 + 2a_5a_6A + b_5a_6 + a_5b_6 + 2Aa_2^2 + 2Aa_3^2 + 2Aa_4^2 + 2a_2b_2 + 2a_3b_3 + 2a_4b_4$. (Here a_i, b_i denote new variables corresponding to the functions $a_i(x_1), b_i(x_1)$, respectively.)

$$T_2 = Q_2 - a_1^2A^2 + b_1^2,$$

where

$Q_2 = (Aa_6 + b_6)^5 + (Aa_6 + b_6)(Aa_5 + b_5) + (Aa_5 + b_5)^6 - 6Aa_5(Aa_5 + b_5)^5 - 5Aa_6(Aa_6 + b_6)^4 - 2A^2a_5a_6 - Ab_5a_6 - Aa_5b_6 - A^2a_2^2 + b_2^2 - A^2a_3^2 + b_3^2 - A^2a_4^2 + b_4^2$.

$$T_3 = 5(a_6A + b_6)^4 + a_5A + b_5.$$

(In particular, Q_1, Q_2, T_3 do not depend on a_1, b_1 .)

Step 8: we could call Algorithm 1 recursively to obtain the output satisfying all the requirements. However, T_1, T_2 are of degree 2 in a_1, b_1 and T_3 is linear with respect to b_5 , therefore, it is easier to compute Nash approximations of a_i, b_i, A by a different direct method, after slight relaxing the requirements regarding the output. Namely, the output polynomials for the approximations

of a_1, b_1 (cf. (b)) will not be monic. Consequently, P_1^ν, P_6^ν corresponding to the Nash approximations f_1^ν, f_6^ν of f_1, f_6 , respectively, will not be monic either but their leading coefficients will be non-vanishing at 0 and their degrees in z_1, z_6 , respectively, will be independent of ν , as required in (b). As before, we are allowed to use only the data representing A, a_i, b_i described in Section 3.2.1 (obtained by applying the effective versions of the Weierstrass Preparation and Division Theorems in Steps 5, 6) and the equations defining the variety (Step 7).

First, using the equation $5(a_6A + b_6)^4 + a_5A + b_5 = 0$ we can eliminate the variable b_5 from Q_1, Q_2 obtaining \tilde{Q}_1, \tilde{Q}_2 , respectively. Now the system

$$\tilde{Q}_1 + 2Aa_1^2 + 2a_1b_1 = \tilde{Q}_2 - a_1^2A^2 + b_1^2 = 0$$

has the following solutions: $a_1 = \frac{-i\tilde{Q}_1}{2\sqrt{\tilde{Q}_1A + \tilde{Q}_2}}$ and $b_1 = \frac{-i(A\tilde{Q}_1 + 2\tilde{Q}_2)}{2\sqrt{\tilde{Q}_1A + \tilde{Q}_2}}$.

We will replace $a_5(x_1), A(x_1), a_j(x_1), b_j(x_1)$, for $j = 2, 3, 4, 6$, in \tilde{Q}_1, \tilde{Q}_2 by polynomial approximations $a_{5,\mu}(x_1), A_\mu(x_1), a_{j,\mu}(x_1), b_{j,\mu}(x_1)$, obtaining $\tilde{Q}_1^\mu, \tilde{Q}_2^\mu$ such that $\frac{i\tilde{Q}_1^\mu}{2\sqrt{\tilde{Q}_1^\mu A_\mu + \tilde{Q}_2^\mu}}$ and $\frac{i(A_\mu\tilde{Q}_1^\mu + 2\tilde{Q}_2^\mu)}{2\sqrt{\tilde{Q}_1^\mu A_\mu + \tilde{Q}_2^\mu}}$ are holomorphic in $D_{\frac{1}{3}}$. To do this, we compute the order κ of zero of $\delta(x_1) := \tilde{Q}_1(x_1)A(x_1) + \tilde{Q}_2(x_1)$ at 0 to confirm that $\kappa = 6$. Since (after Steps 5, 6) we have $Expand_{a_i}, Expand_{b_i}, Expand_A, M_{a_i}, M_{b_i}, M_A$, we also have $Expand_{\delta(x_1)/x_1^6}, M_{\delta(x_1)/x_1^6}$ (where $U_{\delta(x_1)/x_1^6} = D_1$). Therefore, we can check that $\inf_{D_{\frac{1}{3}}} |\delta(x_1)/x_1^6| > 0$ which implies that 0 is the only root of δ in $\overline{D_{\frac{1}{3}}}$. Hence, it is sufficient to approximate a_5, A, a_j, b_j in such a way that 0 is the root of $\tilde{Q}_1^\mu A_\mu + \tilde{Q}_2^\mu$ of order 6 and the root of both numerators of order (at least) 3. Since a_j, b_j, A depend on one variable, this can be done as follows.

In general, one can write $a_j(x_1) = x_1^\kappa \alpha_j(x_1) + p_j(x_1)$, $b_j(x_1) = x_1^\kappa \beta_j(x_1) + q_j(x_1)$, $A(x_1) = x_1^\kappa \gamma(x_1) + e(x_1)$, where p_j, q_j, e are polynomials of degree smaller than κ , and then one can replace $\alpha_j, \beta_j, \gamma$ by polynomial approximations close enough. In the case when 0 is the root of order (at least) 3 of $Aa_j + b_j$ for $j = 2, 3, 4, 6$, (which occurs in our example) one can proceed in a slightly different way: we have

$$\begin{aligned} \tilde{Q}_1 &= -6 \cdot 5^5 a_5 (Aa_6 + b_6)^{20} + a_5 (Aa_6 + b_6) + \sum_{j=2}^4 2a_j (Aa_j + b_j), \\ \tilde{Q}_2 &= -4(Aa_6 + b_6)^5 + 5^6 (Aa_6 + b_6)^{24} + 6 \cdot 5^5 Aa_5 (Aa_6 + b_6)^{20} + \\ &\quad - Aa_5 (Aa_6 + b_6) + \sum_{j=2}^4 -(Aa_j + b_j)(Aa_j - b_j), \\ \tilde{Q}_1 A + \tilde{Q}_2 &= -4(Aa_6 + b_6)^5 + 5^6 (Aa_6 + b_6)^{24} + \sum_{j=2}^4 (Aa_j + b_j)^2. \end{aligned}$$

Therefore, if $A(x_1)a_j(x_1) + b_j(x_1) = x_1^3 \cdot \gamma_j(x_1)$, for some holomorphic γ_j , then it is sufficient to approximate a_5, A, a_j, γ_j , by polynomials $a_{5,\mu}, A_\mu, a_{j,\mu}, \gamma_{j,\mu}$ close enough and define $b_{j,\mu} = -A_\mu(x_1)a_{j,\mu}(x_1) + x_1^3 \cdot \gamma_{j,\mu}(x_1)$, for $j = 2, 3, 4, 6$.

Finally, Nash approximations $a_{1,\mu}, b_{1,\mu}$ of a_1, b_1 and (a polynomial approximation) $b_{5,\mu}$ of b_5 are defined by

$$(3.6) \quad \begin{aligned} a_{1,\mu} &= \frac{-i\tilde{Q}_1^\mu}{2\sqrt{\tilde{Q}_1^\mu A_\mu + \tilde{Q}_2^\mu}}, \quad b_{1,\mu} = \frac{-i(A_\mu\tilde{Q}_1^\mu + 2\tilde{Q}_2^\mu)}{2\sqrt{\tilde{Q}_1^\mu A_\mu + \tilde{Q}_2^\mu}}, \\ b_{5,\mu} &= -5(a_{6,\mu}A_\mu + b_{6,\mu})^4 - a_{5,\mu}A_\mu. \end{aligned}$$

How close $a_{j,\mu}, b_{j,\mu}, A_\mu$ **have to** be to a_j, b_j, A is discussed below. Note that, for P defined above, the method of approximation of A, a_j, b_j presented in Step 8 of this example can be applied for any initial map F for which in Step 5 we have $d = 1$ and for which in Steps 5, 6 we obtain a_1, b_1, A such that $Aa_1 + b_1 \neq 0$. (If $d = 1$, then T_1, T_2, T_3 are of the same form as in Step 7 above. $Aa_1 + b_1 \neq 0$ implies that $\tilde{Q}_1 A + \tilde{Q}_2 \neq 0$. Clearly, the requirement $Aa_1 + b_1 \neq 0$ can be replaced by $Aa_j + b_j \neq 0$ for any $j \in \{1, \dots, 4\}$.)

Step 9: for $j = 2, \dots, 5$, set

$$f_j^\nu(x_1, x_2) = (x_2 - A_\mu(x_1))^2 H_j^\mu(x_1, x_2) + a_{j,\mu}(x_1)x_2 + b_{j,\mu}(x_1),$$

where H_j^μ is a polynomial approximation of H_j close enough. Since $A_\mu, a_{j,\mu}, b_{j,\mu}$ are polynomials, for $j = 2, \dots, 5$, we can define $P_j^\nu(x_1, x_2, z_j) = z_j - f_j^\nu(x_1, x_2)$.

Let us turn to f_1 . Here H_1^μ could also be any polynomial (or Nash) approximation of H_1 . However, using the results of Step 8 (following from T_1, T_2, T_3 being of low degrees in certain variables) one can choose H_1^μ more carefully so that the degree of P_6^ν (obtained in Step 10) in z_6 is relatively small (i.e. equal to 5). We have that $\tilde{Q}_1^\mu A_\mu + \tilde{Q}_2^\mu = x_1^\kappa \tau_\mu(x_1)$, where τ_μ is a polynomial non-vanishing on $D_{\frac{1}{3}}$ (if the approximations performed in Step 8 are close enough). Approximate $H_1 \cdot \sqrt{\tau_\mu}$ by a polynomial η_μ and put $H_1^\mu = \frac{\eta_\mu}{\sqrt{\tau_\mu}}$,

$$f_1^\nu(x_1, x_2) = (x_2 - A_\mu(x_1))^2 H_1^\mu(x_1, x_2) + a_{1,\mu}(x_1)x_2 + b_{1,\mu}(x_1).$$

The order of zero of \tilde{Q}_1^μ and of $(A_\mu \tilde{Q}_1^\mu + 2\tilde{Q}_2^\mu)$ at 0 is at least $\frac{\kappa}{2} = 3$, so $\omega_a(x_1) = \frac{-i\tilde{Q}_1^\mu}{x_1^3}$ and $\omega_b(x_1) = \frac{-i(A_\mu \tilde{Q}_1^\mu + 2\tilde{Q}_2^\mu)}{x_1^3}$ are polynomials. Therefore, we can define

$$P_1^\nu(x_1, x_2, z_1) = 4\tau_\mu z_1^2 - (2\eta_\mu \cdot (x_2 - A_\mu)^2 + \omega_a x_2 + \omega_b)^2.$$

Note that P_1^ν is not monic. More precisely, dividing this polynomial by $4\tau_\mu$, we obtain a monic polynomial but with holomorphic (not polynomial) coefficients (i.e. then the coefficients are rational with denominators not vanishing on $D_{\frac{1}{3}}$).

Step 10: recall that

$$P(f_1^\nu, \dots, f_5^\nu, \bar{f}^\nu) = C^\nu \left(\frac{\partial P}{\partial z_6} (f_1^\nu, \dots, f_5^\nu, \bar{f}^\nu) \right)^2,$$

where $\bar{f}^\nu(x_1, x_2) = (x_2 - A_\mu(x_1))^2 H_6^\mu(x_1, x_2) + a_{6,\mu}(x_1)x_2 + b_{6,\mu}(x_1)$, and H_6^μ is a polynomial approximation of H_6 , and C^ν is a holomorphic function. Moreover, $|C^\nu|$ is small as $a_{j,\mu}, b_{j,\mu}, A_\mu$ are close to a_j, b_j, A , respectively, hence, by Lemma 3.4, there is a Nash function f_6^ν close to \bar{f}_6 with $P(f_1^\nu, \dots, f_5^\nu, f_6^\nu) = 0$. Recall also that

$$V^\nu = \{(x_1, x_2, z) \in \mathbf{C}_{x_1, x_2}^2 \times \mathbf{C}^6 : P(z) = 0, P_i^\nu(x_1, x_2, z_i) = 0 \text{ for } i = 1, \dots, 5\}.$$

Clearly, the graph of f_6^ν is contained in the image $V^{\nu,6}$ of the projection of V^ν to $\mathbf{C}^2 \times \mathbf{C}_{z_6}$. But now we do not know whether $V^{\nu,6}$ is algebraic because P_1^ν is

not monic in z_1 (i.e. the projection need not be proper). Nevertheless, using P_j^ν for $j = 1, \dots, 5$, we can easily eliminate z_1, \dots, z_5 from $P(z_1, \dots, z_6)$ to obtain $P_6^\nu(x_1, x_2, z_6) = \tau_\mu \cdot z_6^5 + \tau_\mu \cdot f_5^\nu \cdot z_6 + \tau_\mu \cdot (f_5^\nu)^6 + \lambda + \tau_\mu \cdot (f_2^\nu)^2 + \tau_\mu \cdot (f_3^\nu)^2 + \tau_\mu \cdot (f_4^\nu)^2$, where $\lambda = \frac{1}{4}(2\eta_\mu \cdot (x_2 - A_\mu)^2 + \omega_a x_2 + \omega_b)^2$. Then $\{P_6^\nu(x_1, x_2, z_6) = 0\}$ contains $V^{\nu,6}$, hence, also the graph of f_6^ν .

Finally, let us estimate how close $a_{j,\mu}, b_{j,\mu}, A_\mu, H_j^\mu$ have to approximate a_j, b_j, A, H_j , for $j = 1, \dots, 6$, to ensure that $\|F - F^\nu\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 10^{-5}$. First, let us follow the proof of Lemma 3.4 to observe that our requirement is satisfied if (3.7)-(3.10) below are fulfilled:

$$(3.7) \quad \|f_j^\nu - f_j\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 10^{-5}, \text{ for } j = 1, \dots, 5,$$

$$(3.8) \quad \|\bar{f}^\nu - f_6\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 5 \cdot 10^{-6},$$

$$(3.9) \quad \|C^\nu\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 10^{-2},$$

$$(3.10) \quad \|2C^\nu \cdot \frac{\partial P}{\partial z_6}(f_1^\nu, \dots, f_5^\nu, \bar{f}^\nu)\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 5 \cdot 10^{-6}.$$

Define $Q \in \mathcal{O}(D_{\frac{1}{3}} \times D_{\frac{1}{3}})[z_6]$ by $Q(z_6) = P(f_1^\nu, \dots, f_5^\nu, z_6)$. By the derivatives of Q we mean the derivatives of $P(f_1^\nu, \dots, f_5^\nu, z_6)$ with respect to z_6 . Since $Q(\bar{f}^\nu) = C^\nu \cdot (Q'(\bar{f}^\nu))^2$, we have (cf. Lemma 1.6 of [34])

$$Q(\bar{f}^\nu + C^\nu \cdot Q'(\bar{f}^\nu) \cdot Y) = C^\nu \cdot (Q'(\bar{f}^\nu))^2 \cdot (1 + Y + \sum_{i=2}^5 (Q^{(i)}(\bar{f}^\nu)/i!) \cdot (Q'(\bar{f}^\nu))^{i-2} \cdot (C^\nu)^{i-1} Y^i).$$

Set

$$\tilde{Q}(Y) = 1 + Y + \sum_{i=2}^5 (Q^{(i)}(\bar{f}^\nu)/i!) \cdot (Q'(\bar{f}^\nu))^{i-2} \cdot (C^\nu)^{i-1} Y^i.$$

Using $Expand_{f_5}, Expand_{f_6}, M_{f_5}, M_{f_6}$ we confirm that $\|f_5\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.2$, $\|f_6\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.01$. Then, by (3.7), (3.8), (3.9) we have: $\|\tilde{Q}(-1)\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \leq \frac{1}{2}$, and $\|Q'(t) - 1\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \leq \frac{1}{4}$, and $\sum_{i=2}^5 \|\tilde{Q}^{(i)}(t)/i!\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \leq \frac{1}{4}$, for every $t \in \mathcal{O}(D_{\frac{1}{3}} \times D_{\frac{1}{3}})$ with $\|t\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \leq 2$. Therefore, there is $y \in \mathcal{O}(D_{\frac{1}{3}} \times D_{\frac{1}{3}})$ with $\|y\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \leq 2$ such that $\tilde{Q}(y) = 0$ (cf. the proof of Lemma 1.5 of [34] with $f = Q$). Thus, $f_6^\nu = \bar{f}^\nu + C^\nu \cdot Q'(\bar{f}^\nu) \cdot y$ satisfies $P(f_1^\nu, \dots, f_5^\nu, f_6^\nu) = 0$, and, by (3.8), (3.10), we have $\|f_6 - f_6^\nu\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 10^{-5}$. So we have proved that it is sufficient to fulfil (3.7)-(3.10).

Now using $Expand_{f_j}, M_{f_j}$, for $j = 5, 6$, confirm that

$$\sup_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \left| \frac{\partial P}{\partial z_6}(f_5, f_6) \right| < 0.2 \text{ and } \inf_{D_{\frac{1}{3}} \times \partial D_{\frac{1}{3}}} \left| \frac{\partial P}{\partial z_6}(f_5, f_6) \right| > 0.13.$$

Therefore, if (3.7), (3.8) hold, then

$$\sup_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} \left| \frac{\partial P}{\partial z_6}(f_5^\nu, \bar{f}^\nu) \right| < 0.21 \text{ and } \inf_{D_{\frac{1}{3}} \times \partial D_{\frac{1}{3}}} \left| \frac{\partial P}{\partial z_6}(f_5^\nu, \bar{f}^\nu) \right| > 0.12,$$

and both (3.9) and (3.10) are satisfied if $\|C^\nu\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 1.25 \cdot 10^{-5}$. In view of the Maximum Principle, it is sufficient to obtain the last inequality on $D_{\frac{1}{3}} \times \partial D_{\frac{1}{3}}$, on which we have

$$C^\nu = \frac{P(f_1^\nu, \dots, f_5^\nu, \bar{f}^\nu)}{\left(\frac{\partial P}{\partial z_6}(f_5^\nu, \bar{f}^\nu)\right)^2}.$$

Since $\inf_{D_{\frac{1}{3}} \times \partial D_{\frac{1}{3}}} \left|\frac{\partial P}{\partial z_6}(f_5^\nu, \bar{f}^\nu)\right| > 0.12$, it is sufficient to require

$$(3.11) \quad \sup_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} |P(f_1^\nu, \dots, f_5^\nu, \bar{f}^\nu)| < 1.8 \cdot 10^{-7}.$$

Using $Expand_{f_j}, M_{f_j}$ we obtain the following bounds: $\|f_1\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.03$, $\|f_2\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.05$, $\|f_3\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.06$, $\|f_4\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.06$; recall that $\|f_5\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.2$, $\|f_6\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 0.01$. These data and the definition of P imply that (3.11) holds if

$$(3.12) \quad \|\bar{f}^\nu - f_6\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 2.8 \cdot 10^{-7}$$

and

$$(3.13) \quad \|f_j^\nu - f_j\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 2.8 \cdot 10^{-7}, \text{ for } j = 1, \dots, 5.$$

Hence, if (3.12), (3.13) are satisfied, then (3.7)-(3.10) are also satisfied.

Now recall that after Step 6,

$$(3.14) \quad f_j(x_1, x_2) = (x_2 - A(x_1))^2 H_j(x_1, x_2) + a_j(x_1)x_2 + b_j(x_1),$$

for $j = 1, \dots, 6$.

Moreover, (cf. Step 9) f_j^ν, \bar{f}^ν are of the form

$$(3.15) \quad f_j^\nu(x_1, x_2) = (x_2 - A_\mu(x_1))^2 H_j^\mu(x_1, x_2) + a_{j,\mu}(x_1)x_2 + b_{j,\mu}(x_1),$$

for $j = 1, \dots, 5$,

and

$$(3.16) \quad \bar{f}^\nu(x_1, x_2) = (x_2 - A_\mu(x_1))^2 H_6^\mu(x_1, x_2) + a_{6,\mu}(x_1)x_2 + b_{6,\mu}(x_1).$$

Confirm that $\|A\|_{D_{\frac{1}{3}}} < 0.03$ and $\|H_j\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 1.5$ for $j = 1, \dots, 6$. These bounds and (3.14)-(3.16) imply that (3.12), (3.13) are satisfied if $\|A - A_\mu\|_{D_{\frac{1}{3}}} < 10^{-7}$, $\|H_j - H_j^\mu\|_{D_{\frac{1}{3}} \times D_{\frac{1}{3}}} < 10^{-7}$, $\|a_j - a_{j,\mu}\|_{D_{\frac{1}{3}}} < 10^{-7}$, $\|b_j - b_{j,\mu}\|_{D_{\frac{1}{3}}} < 10^{-7}$ for $j = 1, \dots, 6$.

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